14. Musical Spider

A tiny spider is sitting on a taut thread of finite length. The thread can vibrate like a string, and has normal modes with frequencies $\omega_k = k\delta$, with $\delta$ some small frequency. The quanta of the normal modes are called phonons. The tiny spider has two states:

$$|\text{ground}\rangle = \left( \begin{array}{c} 0 \\ 1 \end{array} \right), \quad |\text{excited}\rangle = \left( \begin{array}{c} 1 \\ 0 \end{array} \right).$$

The Hamiltonian of the spider has the form

$$H = \begin{pmatrix} E & \epsilon A(t) \\ \epsilon A(t) & 0 \end{pmatrix}; \quad A(t) = \sum_{k \in \mathbb{N}} (a_k e^{-i\omega_k t} + a_k^\dagger e^{i\omega_k t}).$$

$A(t)$ represents the amplitude of the string at the spider’s location; the operators $a_k$ and $a_k^\dagger$ annihilate and create phonons with frequency $\omega_k$, and satisfy $[a_k^\dagger, a_k] = 1$. The parameter $\epsilon$ and the level spacing $\hbar\delta$ are both assumed to be very small compared to $E$.

(a) Suppose that, initially, the thread is in its ground state and the spider is $|\text{excited}\rangle$. Compute the lifetime of $|\text{excited}\rangle$. Verify that your answer has the correct dimension.

(b) Now suppose the thread contains $n$ phonons with the resonant frequency $\omega_{\text{res}} = E/\hbar$. There are now two possible processes, absorption or emission, depending on whether the spider is in the ground state or excited state. Compute the rate for each process.

(c) The thread is warmed up to a finite temperature $T$. What is the expectation value $\langle n_k \rangle$ of the number of phonons with frequency $\omega_k$? Assuming that the spider is in thermal equilibrium with the phonons, find the expectation value of the energy of the spider.

*Hint for (a) and (b): Recall Fermi’s Golden Rule: $R_{i\rightarrow f} = \frac{2\pi}{\hbar} |V_{if}|^2 \rho(E_f).$*
A. Energy band via tunneling

A chain of \( N \) equally spaced identical potential wells with impenetrable potential barriers between them has an \( N \)-fold degenerate ground state. Denote by \( |n\rangle \) the ground state in the \( n \)-th well. The ground state degeneracy is lifted if there is some tunneling between the wells. Let us approximate the Hamiltonian including the tunneling by

\[
H = E_0 + V_0 (T + T^\dagger)
\]

where \( T \) is the operator that shifts one well to the next: \( T|n\rangle = |n + 1\rangle \). We make the periodic identification \( |N + 1\rangle \equiv |1\rangle \).

(a) Suppose the particle sits at \( t = 0 \) in one of the wells. Compute, to leading order in \( V_0 \), the probability that at time \( t \) it sits in the next well.

(b) Find the eigenstates of \( T \). Show that these are also eigenstates of \( H \), and compute the eigenenergies.

(c) Discuss the limit \( N \to \infty \). Show that the tunneling splits the infinitely degenerate ground state into a continuous energy band. What is the maximum energy in the band?

B. Spin in rotating B-field

A spin 1/2 particle is placed in a magnetic field: \( \mathbf{B}_0 = -B_0 \mathbf{z} \). The initial spin state is \( |\uparrow\rangle \).

At \( t = 0 \), another weak magnetic field \( B_1 \ll B_0 \) is turned on. This field \( \mathbf{B}_1 \) starts in the \( x \)-direction, but rotates with constant angular frequency \( \omega \) around the \( z \)-axis:

\[
\mathbf{B}_1 = B_1 (\cos(\omega t) \mathbf{x} - \sin(\omega t) \mathbf{y})
\]

(a) Calculate, to first order in \( B_1 \), the probability that the particle is in the state \( |\downarrow\rangle \) at time \( t \). What is the resonant frequency?

*Hint:* The Hamiltonian for a spin 1/2 in a B-field is given by \( H = \mu_\mu \mathbf{B} \cdot \mathbf{\sigma} \).

**Bonus question:** Can you find the *exact* transition probability, to all orders in \( B_1 \)?