Physics 305
Problem Set 1
Due at lecture on Thursday, Sept. 22, 2005

1. Spin 1/2 in a magnetic field

The two-dimensional space of states of a spin 1/2 particle at rest, spanned by |s=1/2, s_z=±1/2⟩, is well suited to illustrate the postulates of quantum mechanics. We choose a basis of the Hilbert space as follows

\[
\begin{pmatrix}
1 \\
0
\end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{en} \quad \begin{pmatrix}
0 \\
1
\end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.
\]

(a) Express s_x, s_y en s_z in terms of Pauli matrices and find the eigenvalues and eigenstates of each of these operators.

(b) A measurement of s_x gives s_x = +\frac{1}{2}\hbar. What is the state of the system immediately after this measurement? We now immediately measure s_z. What is the probability of each possible outcome? Compute the expectation value \langle s_z \rangle.

(c) We turn on a magnetic field in the z direction. The resulting Hamiltonian of the spin 1/2 particle at rest reads \( H = \mu_B B \sigma_z \). Give the general solution to the Schrödinger equation.

(d) At \( t = 0 \), the system is in the eigenstate of \( s_x \) with eigenvalue \( +\frac{1}{2}\hbar \). It then evolves via the Hamiltonian given in part (c). At \( t = t_1 > 0 \) we measure \( s_x \). What is the probability of each possible outcome?

(e) What if instead we measure \( s_z \) at \( t = t_1 \)? Explain the characteristic difference with your answer in (d).

2. Phonons in one dimension

As a toy model for a one-dimensional solid, let us consider a linear chain of masses \( m \), connected with springs with spring constant \( \kappa \). When in equilibrium, the masses form a regular one-dimensional lattice. Now let \( x_i \) denote the displacement of the \( i \)-th mass from its equilibrium position, and let \( p_i \) be the corresponding momentum. We further assume that there are \( N \) masses and impose periodic boundary conditions: \( x_{i+N} = x_i \). The Hamiltonian for this system of masses is

\[
H_0 = \sum_{i=1}^{N} \left( \frac{p_i^2}{2m} + \frac{1}{2}m\omega^2 x_i^2 + \frac{1}{2}\kappa(x_i - x_{i-1})^2 \right),
\]
(a) Show that $H$ is invariant under the cyclic permutation
\[ T_1 : \ (x_i, p_i) \rightarrow (x_{i+1}, p_{i+1}) . \]

Why is this result useful? What are the possible eigenvalues of $T_1$?

It is convenient to combine the $N$ positions and momenta into $N$ component vectors
\[ \mathbf{x} := \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix} \quad \text{and} \quad \mathbf{p} := \begin{pmatrix} p_1 \\ \vdots \\ p_N \end{pmatrix} , \]

In this notation, $T_1$ is given by the $N \times N$ matrix
\[ T_1 := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 1 & \cdots & \cdots & 0 & 1 \end{pmatrix} . \]

(b) Show that the hamiltonian in this vector notation takes the form
\[ H = \frac{\mathbf{p}^T \mathbf{p}}{2m} + \frac{1}{2} m \omega^2 \mathbf{x}^T \mathbf{x} + \frac{1}{2} \kappa \mathbf{x}^T \mathbf{M} \mathbf{x} \]
with (here $\mathbf{I}$ denotes the $N \times N$ identity matrix)
\[ \mathbf{M} = 2\mathbf{I} - T_1 - T_1^T \]

(c) Find the normalized eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_N$, of $T_1$. Show that
\[ \mathbf{M} \mathbf{v}_k = \lambda_k \mathbf{v}_k , \quad \lambda_k = 2 - 2 \cos \left( \frac{2\pi k}{N} \right) \]
We now define the unitary $N \times N$ matrix $V = (v_1, \ldots, v_N)$, and introduce new position and momentum operators via

$$\tilde{x} := V^\dagger x \quad \text{and} \quad \tilde{p} := Vp.$$ 

(d) Verify that the new position and momentum operators $\tilde{x}_i$ and $\tilde{p}_i$ satisfy the standard commutation relations $[\tilde{x}_i, \tilde{p}_j] = i\hbar\delta_{ij}$.

(e) Show that

$$H = \frac{\tilde{p}^\dagger \tilde{p}}{2m} + \frac{1}{2} m \omega^2 \tilde{x}^\dagger \tilde{x} + \frac{1}{2} \kappa \tilde{x}^\dagger D \tilde{x}$$

with

$$D = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_N \end{pmatrix}. $$

(f) Rewrite $H$ in terms of raising and lowering operators:

$$H = \sum_{i=1}^N \hbar \omega_i (a_i^\dagger a_i + \frac{1}{2})$$

with $[a_i, a_j] = 0$ and $[a_i, a_j^\dagger] = \delta_{ij}$. Determine the eigenfrequencies $\omega_i$. Make a plot of $\omega_i$ as a function of $i$ for $N = 20$.

(g) Describe the spectrum and eigenstates of $H$.

The operators $a_i^\dagger, a_i$ are said to create phonons. We say that a state $|\psi\rangle$ with

$$a_i^\dagger a_i |\psi\rangle = N_i |\psi\rangle$$

has $N_i$ phonons of frequency $\omega_i$. Phonons are the quanta of lattice vibrations. (They are analogous to photons, which are the quanta of oscillations of the electro-magnetic field.)

(h) Compute the expectation values $\langle x_k(t) \rangle$ and $\langle x_k^2(t) \rangle$, with $x_k(t)$ the displacement at time $t$ of the $k$-th mass, in the state $|\psi\rangle$ defined above, with $N_i$ phonons of frequency $\omega_i$ (and zero phonons of frequency $\omega_k$ for all $k \neq i$).

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3. Oscillating neutrino

A neutrino can interact with matter as an electron-neutrino $\nu_e$ or as a muon-neutrino $\nu_{\mu}$. We therefore write its wave function as $|\psi\rangle = a|\nu_e\rangle + b|\nu_{\mu}\rangle$. The Hamiltonian is given by (we set $c = 1$)

$$H = p + \frac{M^2}{2p}$$

where $M^2$ is a $2 \times 2$ matrix with eigenvectors

$$|\nu_1\rangle = \cos \theta |\nu_e\rangle + \sin \theta |\nu_{\mu}\rangle$$

$$|\nu_2\rangle = \cos \theta |\nu_{\mu}\rangle - \sin \theta |\nu_e\rangle,$$

with respective eigenvalues $m_1^2$ and $m_2^2$.

At $t = 0$, a neutrino with momentum $p$ is in the state $|\nu_e\rangle$. At time $t$ it reacts with matter inside of a detector. What is the probability that it is detected as a muon-neutrino?

4. Particles in a 1-D Box

We first consider a single particle of mass $m_1$ that moves freely along the $x$-axis between hard walls at $x = \pm a$. As is standard, label the energy eigenstates counting upwards in energy from the ground state as $n = 1, 2, 3, \ldots$. Assume all motion is non-relativistic.

(a) For each energy eigenstate, give the eigenenergy $E_n$ and normalized wavefunction $\psi_n(x)$.

When the size $a$ of the box is very large, the spectrum of all eigenenergies is very closely spaced. A useful quantity is the density of states $\rho(E)$, defined such that

$$\rho(E)\Delta E = \text{number of energy eigenstates between } E \text{ and } E + \Delta E.$$

(b) Show that the density of states of the particle in a large box is given by

$$\rho(E) = \frac{a}{\pi \hbar} \sqrt{\frac{2m}{E}}.$$
(Hint: use that \( \rho(E) = \frac{dN(E)}{dE} \) with \( N(E) \) the number of energy levels below \( E \).)

Now also put a second particle with mass \( m_2 \neq m_1 \) in the same box. Assume the two particles do not interact with each other. The energy eigenstates of this two-particle system can be labeled by the energy quantum numbers of each particle \((n_1, n_2)\); the corresponding two-particle wavefunction is simply the product of the single-particle wavefunctions you obtained in part (a):

\[
\Psi_{n_1n_2}(x_1, x_2) = \psi_{n_1}(x_1)\psi_{n_2}(x_2).
\]

This state may also be denoted by the Dirac ket \( |n_1, n_2\rangle \).

(c) What is the spectrum (all the energy eigenvalues \( E_{n_1n_2} \)) of this two-particle system?

Now let \( m_1 = m_2 \). The system now has a discrete symmetry: that of interchanging the positions of the two particles. In other words: the operator that interchanges the positions of the two particles commutes with the Hamiltonian. Assume for now that the particles remain distinguishable.

(d) Give a complete set of normalized states of the two-particle system that are eigenstates of both the Hamiltonian and the position-interchange operator. You may give these as wavefunctions \( \Psi(x_1, x_2) \) written in terms of the \( \psi_n \) you obtained in part (a) above, or more abstractly as states written in terms of the kets \( |n_1, n_2\rangle \). What pairs of states are degenerate (in eigenenergy) due to the \( m_1 = m_2 \) symmetry?

If the particles are instead indistinguishable bosons, then only states that are symmetric under all pairwise interchanges of all particle labels are physical.

(e) If there are \( N \) indistinguishable bosons in the box, what is the degeneracy (if any) of the lowest-energy and next-to-lowest-energy excited states of the system?

If the particles are indistinguishable fermions, then only states that are antisymmetric under all pairwise interchanges of all particle labels are physical.

(f) Show that fermions obey the Pauli exclusion principle: no two particles can be in the same state. If there are \( N \) indistinguishable fermions in the box, what is the ground state of the system? What is the ground state energy?

5. Ring around flux

Consider a particle with charge \( e \) and mass \( m \) confined to a circular ring with radius \( R \). Suppose the ring surrounds a magnetic flux \( \Phi \) but the magnetic field vanishes everywhere on and near the ring. Find the energy levels of the particle on the ring and for each level find the corresponding electric current. Treat the motion along the ring as a one-dimensional problem.
6. Superstring theory

A string has normal modes with frequencies $\omega_n = \gamma n$, with $n$ any positive integer. Each normal mode is a harmonic oscillator with energy levels $N_n \hbar \omega_n$. (It is customary to drop the ground state contribution $\frac{1}{2} \hbar \omega_n$.) Let us call the elementary excitations of the string phonons. Phonons are bosons, and $N_n$ counts the number of phonons with frequency $\omega_n$.

(a) Determine the degeneracy of the lowest five energy levels of the string.

A fermionic string also has phonons, but these are now fermions – so let’s call them phoninos. They come in two types, spin $\uparrow$ or spin $\downarrow$, and have half-integer frequencies: $\omega_n = \gamma (n + \frac{1}{2})$ with $n \geq 0$. The total spin of all phoninos must add up to zero.

(b) Determine the degeneracy of the lowest five energy levels of this fermionic string. *Hint:* You should find the same spectrum and degeneracies as in part (a).

7. Emitted photon

An electron in a hydrogen atom is in an excited state with $l = 0$. It decays to a lower energy state, while emitting a photon. The energy difference between the initial and final state is $E$ and the linewidth of the transition is $\Delta E$. The atom and photon are inside of a box with volume $V$. Estimate the total number of possible final states of the photon.

8. Electron in a constant magnetic field

We consider an electron moving in the $xy$-plane, in a uniform magnetic field in the $z$-direction, $\mathbf{B} = B\mathbf{e}_z$.

(a) Verify that the Lorentz force $\mathbf{F}_{\text{Lorentz}}$ indeed always lies in the $xy$-plane.

The Hamiltonian of the electron is given by

$$ H = \frac{1}{2m} (\mathbf{p} - eA(x))^2 $$
For the vector potential we choose the gauge

\[ \vec{A} = -B \begin{pmatrix} y \\ 0 \\ 0 \end{pmatrix}. \]

(b) Derive the equation of motion of the position expectation value. Verify that it corresponds with the Lorentz force. In other words, show that:

\[ \frac{d^2}{dt^2} \langle \vec{x} \rangle = \langle \vec{F}_{\text{Lorentz}} \rangle. \]

(c) Show that the momentum \( p_x \) in the \( x \)-direction is conserved. Is the velocity in the \( x \)-direction also conserved?

(d) Verify that eigenstates \( |\psi_{k_x}\rangle \) of \( p_x \) in the position representation are given by wavefunctions \( \psi_{k_x}(\vec{x}) = \langle \vec{x} | \psi_{k_x} \rangle \) of the form

\[ \psi_{k_x}(\vec{x}) = e^{ik_x x} \phi(y). \]

(e) Show that \( H \) acting on \( \psi_{k_x}(\vec{x}) \) reduces (after a simple redefinition of the \( y \)-coordinate) to the Hamiltonian of a simple harmonic oscillator

\[ H = \frac{1}{2m} \left( \frac{\hbar}{i} \frac{\partial}{\partial y} \right)^2 + \frac{1}{2} m \omega^2 y^2. \]

Give the expression for the frequency \( \omega \).

(f) Give all eigenfunctions and eigenvalues of \( H \). Explain why the energy eigenvalues do not depend on \( k_x \). Give the probability distribution \( |\psi_{k_x}(\vec{x})|^2 \) of a state with lowest energy.

We restrict the \( xy \)-plane to a strip of length \( L \) in the \( x \)-direction en width \( W \) in the \( y \)-direction. We ignore boundary conditions in the \( y \)-direction and assume that the wavefunctions obtained in part (e) remain valid. We place several electrons inside of this strip, and assume that they only interact via the Pauli exclusion principle.

(g) What are the possible values of \( k_x \)? Estimate the maximal number of electrons that fits in the lowest energy level and the electron density when all lowest energy levels are filled.