Physics 305: Summary of time-dependent perturbation theory

We wish to solve the time-dependent Schrödinger equation of a Hamiltonian $H = H^0 + H^1(t)$ with a small time-dependent perturbation:

$$i\hbar \frac{d}{dt} |\Psi_s\rangle = (H^0 + H^1(t)) |\Psi_s\rangle.$$  \hspace{1cm} (1)

It will be useful to write the wave function and observables in the Dirac picture. These are related to the Schrödinger picture via

$$|\Psi_D\rangle = e^{\frac{i}{\hbar}H^0t} |\Psi_s\rangle, \hspace{1cm} A_D = e^{\frac{i}{\hbar}H^0t} A_s e^{-\frac{i}{\hbar}H^0t}.$$  \hspace{1cm} (2)

The two pictures are evidently equivalent, since

$$\langle \Psi_D | A_D | \Psi_D \rangle = \langle \Psi_s | A_s | \Psi_s \rangle.$$  \hspace{1cm} (3)

The Schrödinger equation for the Dirac picture wave function reads

$$i\hbar \frac{d}{dt} |\Psi_D\rangle = H^1_D(t) |\Psi_D\rangle.$$  \hspace{1cm} (4)

where

$$H^1_D(t) = e^{\frac{i}{\hbar}H^0t} H^1(t) e^{-\frac{i}{\hbar}H^0t}.$$  \hspace{1cm} (5)

The advantage of this formulation is that the time-dependence of $|\Psi_D\rangle$ is generated by the small perturbation $H^1$.

We can formally solve (4) by writing

$$|\Psi_D\rangle(t) = U(t) |\Psi_D\rangle(0)$$  \hspace{1cm} (6)

where the operator $U(t)$ is a unitary operator that satisfies

$$i\hbar \frac{d}{dt} U(t) = H^1_D(t) U(t) \hspace{1cm} U(0) = 1.$$  \hspace{1cm} (7)

$U(t)$ is called the evolution operator. To first order in perturbation theory it is given by

$$U(t) \simeq 1 - \frac{i}{\hbar} \int_0^t dt' H^1_D(t').$$  \hspace{1cm} (8)
To proceed, we expand in energy eigen functions of the unperturbed Hamiltonian $H_0$

$$|\Psi_0(t)\rangle = \sum_n c_n(t) |n\rangle, \quad H^0 |n\rangle = E_n |n\rangle. \quad (9)$$

Let us assume that at $t = 0$, the system is in the eigen state $|n\rangle$:

$$c_n(0) = 1, \quad \text{while} \quad c_m(0) = 0, \quad \text{for all} \quad m \neq n \quad (10)$$

At a later time $t$, we can write $|\Psi_0\rangle = U(t)|n\rangle$, which using (8) becomes

$$|\Psi_0\rangle(t) = \left(1 - \frac{i}{\hbar} \int_0^t dt' H_0^1(t')\right) |n\rangle \quad (11)$$

From this we can now immediately obtain the coefficients $c_m = \langle m | \Psi_0 \rangle$ at time $t$: we just need to take the inner product of (11) with $\langle m |$:

$$c_m(t) = -\frac{i}{\hbar} \int_0^t dt' \langle m | H_0^1(t) | n \rangle \quad (12)$$

Finally, we re-express the matrix element of $H_0^1(t)$ in terms of the Schrödinger operator $H^1(t)$ via

$$\langle m | H_0^1(t) | n \rangle = e^{i(E_m - E_n)t} \langle m | H^1(t) | n \rangle, \quad (13)$$

where we used (5). This results in the final answer for $c_m(t)$:

$$c_m(t) = -\frac{i}{\hbar} \int_0^t dt' e^{i(E_m - E_n)t} \langle m | H^1(t) | n \rangle$$

The transition probability, that is, the probability for the system at time $t$ to be in the state $|m\rangle$ given that at time $t = 0$ it is in the state $|n\rangle$ is

$$P_{n-m}(t) = |c_m(t)|^2 \quad (15)$$