1. **Superstring theory** [30 points]

A string has normal modes with frequencies \( \omega_n = \gamma n \), with \( n \) any positive integer. Each normal mode is a harmonic oscillator with energy levels \( N_n \hbar \omega_n \). (It is customary to drop the ground state contribution \( \frac{1}{2} \hbar \omega_n \).) Let us call the elementary excitations of the string **phonons**. Phonons are bosons, and \( N_n \) counts the number of phonons with frequency \( \omega_n \).

(a) Determine the degeneracy of the lowest five energy levels of the string. [15 points]

A fermionic string also has phonons, but these are now **fermions** – so let’s call them **phoninos**. They come in two types, spin \( \uparrow \) or spin \( \downarrow \), and have half-integer frequencies: \( \omega_n = \gamma (n + \frac{1}{2}) \) with \( n \geq 0 \). The total spin of all phoninos must add up to zero.

(b) Determine the degeneracy of the lowest five energy levels of this fermionic string. **Hint:** You should find the same spectrum and degeneracies as in part (a). [15 points]

A state with two phoninos:

- one with spin \( \uparrow \) and \( E = \frac{1}{2} \hbar \gamma \)
- one with spin \( \downarrow \) and \( E = \frac{3}{2} \hbar \gamma \)

\[
\begin{array}{c}
7/2 \hbar \gamma \\
5/2 \hbar \gamma \\
3/2 \hbar \gamma \\
1/2 \hbar \gamma \\
\end{array}
\]

E
2. Two spinning spins [40 points]

Two spin $\frac{1}{2}$ particles have fixed positions, and their spins interact via the Hamiltonian

$$H = -J \vec{S}_A \cdot \vec{S}_B.$$ 

At time $t = 0$, spin $A$ points in the positive $z$-direction and spin $B$ in the negative $z$-direction. We want to compute the state of spin $A$ at time $t$.

(a) Write $H$ in terms of the operators $(S_A)^2$, $(S_B)^2$ and $(S_A + S_B)^2$. What are the eigenvalues of $H$? Give the energy eigenstates. [15 points]

(b) Show that the state of the two spins at time $t$ is of the form

$$|\psi(t)\rangle = e^{-i\omega t} \left( \cos \omega t |\uparrow\rangle_A |\downarrow\rangle_B + i \sin \omega t |\downarrow\rangle_A |\uparrow\rangle_B \right)$$

Compute $\alpha$ and $\omega$. [10 points]

(c) Compute the density matrix of spin $A$ at time $t$. At which times $t$ does it describe a pure state, that is, at which times does the entanglement between the two spins vanish? [15 points]

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Initial situation of problem 3:

![Initial situation of problem 3](image-url)
3. Exciting collision  [60 points]

Consider two particles of mass \(m\) moving in one dimension. Particle 1 moves freely, while particle 2 experiences a harmonic potential \(V(x_2) = \frac{1}{2}m\omega^2 x_2^2\). The two particles interact via a delta-function potential

\[ V_{\text{int}}(x_{12}) = \lambda \delta(x_{12}), \]

with \(x_{12} = x_1 - x_2\). Particle 2 starts in the ground state \(|\psi_0\rangle\), and particle 1 comes in from the left in a momentum eigenstate \(|p_i\rangle\). We want to compute the transition probability \(\mathcal{P}_{01}\) that particle 2 ends up in the first excited state \(|\psi_1\rangle\), to leading order for small \(\lambda\).

(a) Assuming that particle 2 indeed gets excited via the collision with particle 1, determine the final energy \(E_f\) and momentum \(p_f\) of particle 1.  

To proceed, it is useful to put particle 1 in a one-dimensional periodic box of some large size \(L\). Consequently, the collision between the two particles will be a recurring event. The resulting transition rate \(\mathcal{R}_{01}\) for exciting particle 2 can be computed via Fermi’s Golden rule

\[ \mathcal{R}_{01} = \frac{2\pi}{\hbar} |V_{if}|^2 \rho(E_f) \]

where \(V_{if} = \langle \psi_1 | (p_f | V_{\text{int}} | p_i) | \psi_0 \rangle\).

(b) Explain why the transition probability \(\mathcal{P}_{01}\) per collision is related to the rate \(\mathcal{R}_{01}\) by

\[ \mathcal{P}_{01} = \frac{mL}{p_i} \mathcal{R}_{01} \]

(c) Compute the density of states \(\rho(E_f)\) per unit energy for particle 1.

(d) Show that:

\[ \langle p_f | V_{\text{int}} | p_i \rangle = \frac{\lambda}{L} e^{\frac{(p_i - p_f)^2}{2m\omega}} \]

(e) Compute \(V_{if}\) and the transition probability \(\mathcal{P}_{01}\). The answer you should find is

\[ \mathcal{P}_{01} = \frac{\lambda^2 m (p_i - p_f)^2}{\hbar^3 \omega p_i p_f} e^{-\frac{(p_i - p_f)^2}{2mk^2}} \]

(f) Verify that the above final answer has the right dimension.
4. **Scattering off twin potential** [30 points]

(a) Give the definition of the differential cross-section $d\sigma/d\Omega$. [5 points]

(b) Consider two identical potentials mutually displaced by a vector $2\vec{a} = (0, 0, 2a)$:

$$V(\vec{r}) = V_0(\vec{r} - \vec{a}) + V_0(\vec{r} + \vec{a})$$

Using the first Born approximation, express the amplitude $f(\theta)$ and differential cross section $d\sigma/d\Omega$ for scattering off this potential $V(\vec{r})$ in terms of the scattering amplitude $f_0(\theta)$ of one potential $V_0(\vec{r})$. Assume the incoming beam consists of particles of mass $m$ and momentum $\vec{p} = (0, 0, \hbar k)$. Discuss the limiting cases $ka \ll 1$ and $ka \gg 1$. Give a qualitative physical interpretation of the result in terms of interference phenomena. [25 points]