Problem 11. Periodic potential
As a simple model of an electron moving in a crystal, consider a single quantum particle moving (nonrelativistically) in one dimension in the sinusoidal potential

\[ V(x) = V_0 \cos(Kx). \]

Treat this potential as a perturbation to the free particle, using the basis of momentum eigenstates. It is convenient to use “periodic boundary conditions,” which amounts to wrapping the line around into a circle, with \( N \) periods of the potential around the circle. We are interested in the limit where \( N \) is large.

(a) For a state with generic momentum \( \hbar k \) there are no degeneracies. Using non-degenerate perturbation theory, solve for the eigenenergy to second order in \( V_0 \) and the eigenfunction \( \psi(x) \) to first order in \( V_0 \). For what ranges of \( k \) and \( V_0 \) are these results reasonable approximations?

(b) For \( k \) near enough to \( \pm K/2 \), the near-degeneracy of the two states with \( k \) near \( \pm K/2 \) means that simple non-degenerate perturbation theory is not appropriate. For these cases obtain the eigenenergies and eigenfunctions in the approximation where you include only the contribution from these two near-degenerate eigenstates, so the Hamiltonian may be expressed as a 2 \( \times \) 2 matrix. Confirm that your results match well to those of part (a) for small \( V_0 \) as \( k \) moves far enough away from \( \pm K/2 \). What is the magnitude of the gap in the spectrum of this particle’s Hamiltonian?

Problem 12. Shaken oscillator
Consider a single harmonic oscillator with frequency \( \omega \) and a small time-dependent perturbation

\[ H = \frac{\hbar^2}{2m} + \frac{m\omega^2 x^2}{2} + \epsilon(t) x \]

with

\[\begin{align*}
\epsilon(t) &= 0 & t < 0 \\
&= \epsilon & 0 < t < T \\
&= 0 & T < t
\end{align*}\]

The particle starts out in the ground state.
(a) Compute the expectation value of $x$ as a function of time $t < T$.

(b) Using first order perturbation theory, find the transition probability for the particle to end up in the first excited state. Explain what happens for $T = \frac{2\pi}{\omega}$.

13. Einstein A en B coefficients

Consider a system that consists of atoms with two energy levels $E_1$ and $E_2$ and a thermal gas of photons. There are $N_1$ atoms with energy $E_1$, $N_2$ atoms with energy $E_2$ and the energy density of photons with frequency $\omega = (E_2 - E_1)/\hbar$ is $W(\omega)$. In thermal equilibrium at temperature $T$, $W$ is given by the Planck distribution:

$$ W(\omega) = \frac{\hbar \omega^3}{\pi^2 c^3 \exp(h\omega/k_B T) - 1}. $$

According to Einstein, this formula can be understood, by assuming the following rules for the interaction between the atoms and the photons

- Atoms with energy $E_1$ can absorb a photon and make a transition to the excited state with energy $E_2$; the probability per unit time for this transition to take place is proportional to $W(\omega)$, and therefore given by

  $$ P_{\text{abs}} = B_{12} W(\omega) $$

  for some constant $B_{12}$.

- Atoms with energy $E_2$ can make a transition to the lower energy state via stimulated emission of a photon. The probability per unit time for this to happen is

  $$ P_{\text{stim}} = B_{21} W(\omega) $$

  for some constant $B_{21}$.

- Atoms with energy $E_2$ can also fall back into the lower energy state via spontaneous emission. The probability per unit time for spontaneous emission is independent of $W(\omega)$. Let’s call this probability

  $$ P_{\text{spont}} = A_{21}. $$

$A_{12}$, $B_{12}$ and $B_{21}$ are known as Einstein coefficients.

(a) Write a differential equation for the time dependence of the occupation numbers $N_1$ and $N_2$. 

2
(b) What is the lifetime of the excited energy level $E_2$ at very low temperature?

(c) Determine the distribution $W(\omega)$ in thermal equilibrium as a function of the Einstein coefficients.

Assume that the ratio $N_1/N_2$ in thermal equilibrium is given by the Boltzmann factor

$$\frac{N_1}{N_2} = \exp(\hbar\omega/k_B T).$$

(d) By comparing the result of part (c) with the Planck distribution, show that

$$P_{abs} = P_{stim} = \langle n \rangle P_{spont}$$

where $\langle n \rangle = 1/(\exp(\hbar\omega/k_B T) - 1)$ is the average number of photons with frequency $\omega$. Give an interpretation of this formula. When is spontaneous emission dominant, when stimulated?