1. Particle tunneling through barrier.

We first look for the lowest two energy eigen states. We will assume that

\[ \epsilon^2 \equiv \frac{\hbar^2}{2ma^2v_0} \ll 1. \tag{1} \]

The two (respectively even and odd) wave functions \( \Psi_{\pm}(x) \) are good WKB energy eigen functions, that satisfy all boundary and continuity conditions, provided that (for the case of \( \Psi_+ \))

- continuity of \( \Psi_+ \) at \( x = \pm a \):
  \[ 2A \sin(k_+a) = 2B \cosh(\phi/2) \]
- continuity of \( \Psi'_+ \) at \( x = \pm a \):
  \[ -2Ak_\pm \cos(k_+a) = 2B\frac{|p(a)|}{\hbar} \sinh(\phi/2) \]

with

\[ \phi = \frac{1}{\hbar} \int_{-a}^{a} |p(x')| \, dx' \]

Taking the ratio of both sides of these two equations, and using

\[ \frac{|p(a)|}{\hbar} = \sqrt{\frac{2mv_0}{\hbar^2}} = \frac{1}{a\epsilon} \]

gives:

\[ ak_+ \cot(k_+a) = -\frac{1}{\epsilon} \tanh(\phi/2) \]

The same calculation for the \( \Psi_- \) gives:

\[ ak_- \cot(k_-a) = -\frac{1}{\epsilon} \coth(\phi/2) \]

(b) The WKB assumption is that \( \phi \) is large, so \( e^\phi \) is extremely large. Now since

\[ \tanh(\phi/2) = \frac{1 - e^{-\phi}}{1 + e^{-\phi}} \simeq 1 - 2e^{-\phi} \]

\[ \coth(\phi/2) = \frac{1 + e^{-\phi}}{1 - e^{-\phi}} \simeq 1 + 2e^{-\phi} \]

are both very close to 1, the difference between \( k_+ - k_- \) is indeed very small.
The eqns (4) determine the wave numbers $k_\pm$ of $\Psi_\pm$ in terms of known quantities, $\epsilon$ and $\phi$. When $\epsilon$ is small and $e^\phi$ is very large, the solution is well approximated by

$$k_\pm \simeq \frac{\pi}{a}(1 - \epsilon(1 + 2e^{-\phi})), \quad (2)$$

(c) The difference $\Delta E$ between the eigen energies $E_\pm$ of $\Psi_\pm$ is given by

$$\Delta E = \frac{\hbar^2}{2m}(k_+^2 - k_-^2) \simeq \frac{4\hbar^2\pi^2 \epsilon e^{-2\phi}}{ma^2}.$$ 

(d) The state at time $t$ is given by

$$\Psi(t) = \frac{1}{\sqrt{2}}(e^{-\frac{i}{\hbar}(E_+ + E_-)t}\Psi_+ + e^{-\frac{i}{\hbar}(E_+ + E_-)t}\Psi_-)$$

$$= e^{-\frac{i}{2}(E_+ + E_-)t}(\cos(\omega t)\Psi_R + \sin(\omega t)\Psi_L)$$

with

$$\Psi_{L,R} = \frac{1}{\sqrt{2}}(\Psi_+ \pm \Psi_-)$$

and

$$\omega = \frac{1}{2}\Delta E$$

So the probability to be in the left state is

$$P(t) = \sin^2 \omega t.$$ 

(e) To verify the approximation, one uses that

$$\cot(\pi - \delta) \simeq \frac{1}{\delta}$$

and the equations given in the solution to part (b).

2. Two spinning spins

(a) We can write:

$$H = -\frac{J}{2}((S_A + S_B)^2 - (S_A)^2 - (S_B)^2).$$

The rules for adding angular momentum give $(S_A + S_B)^2 = \hbar^2 j(j+1)$ with $j = 0$ or $j = 1$, while the last two terms are both equal to $3\hbar^2/4$. So we have

$$H = -\frac{J}{2}\hbar^2(j(j+1) - 3/2) = \begin{cases} \frac{3}{4}J\hbar^2, & j = 0 \quad \text{(singlet)} \\ -\frac{1}{4}J\hbar^2, & j = 1 \quad \text{(triplet)} \end{cases}. $$
The singlet state is
\[ |j = 0\rangle = \frac{1}{\sqrt{2}} (|\uparrow\rangle_A |\downarrow\rangle_B - |\downarrow\rangle_A |\uparrow\rangle_B) \]
and the triplet states are
\[ |j = 1, m = +1\rangle = |\uparrow\rangle_A |\uparrow\rangle_B, \quad |j = 1, m = -1\rangle = |\downarrow\rangle_A |\downarrow\rangle_B, \]
\[ |j = 1, m = 0\rangle = \frac{1}{\sqrt{2}} (|\uparrow\rangle_A |\downarrow\rangle_B + |\downarrow\rangle_A |\uparrow\rangle_B) \]

The initial state is the sum of the singlet state and \( m = 0 \) triplet state. After a time \( t \), each energy eigenstate is multiplied by \( e^{-\frac{i}{\hbar}E_j t} \). So
\[ |\psi(t)\rangle = \frac{1}{\sqrt{2}} \left( e^{-\frac{i}{4}\hbar J t} |j = 0\rangle + e^{\frac{i}{4}\hbar J t} |j = 1, m = 0\rangle \right) \]

We can rewrite this as
\[ |\psi(t)\rangle = e^{-\alpha t} \left( \cos \omega t |\uparrow\rangle_A |\downarrow\rangle_B + i \sin \omega t |\downarrow\rangle_A |\uparrow\rangle_B \right) \]
with \( \alpha = \frac{\hbar}{4} J \) and \( \omega = \frac{\hbar}{2} J \).

(b) To obtain the density matrix of spin \( A \) we have to take the trace of \( \rho = |\psi\rangle \langle \psi| \) over the Hilbert space of spin \( B \):
\[ \rho_A = \sum_{s=\downarrow,\uparrow} \langle s | \psi \rangle \langle \psi| s \rangle_B \]
where \( |\psi\rangle \) is the state given in (\*). A simple calculation gives
\[ \rho_A = \cos^2 \omega t |\uparrow\rangle \langle \uparrow| + \sin^2 \omega t |\downarrow\rangle \langle \downarrow| \]
This shows that the probability of finding spin \( A \) in the spin up (resp. down) state at time \( t \) is \( \cos^2 \omega t \) (resp \( \sin^2 \omega t \)). The identity \( \text{tr} \rho_A = \cos^2 \omega t + \sin^2 \omega t = 1 \) tells us that probabilities indeed add up to 1. On the other hand:
\[ \text{tr} \rho_A^2 = \cos^4 \omega t + \sin^4 \omega t < 1 \]
except when \( \cos \omega t = \pm 1 \) or \( \sin \omega t = \pm 1 \). So spin \( A \) is in a mixed state, except for
\[ t = \frac{\pi n}{2\omega} = \frac{\pi J n}{\hbar}, \quad \text{with } n \text{ integer}. \]
3. Scattering

The differential cross section is

$$\frac{d\sigma}{d\Omega} = |f(\theta, \varphi)|^2$$  \hspace{1cm} (3)

In the spherically symmetric potential there is no dependance on $\varphi$. In the first Born approximation we have

$$f(\theta) = -\frac{2m}{\hbar^2\kappa} \int_{0}^{\infty} V(r) \sin(\kappa r) r dr$$  \hspace{1cm} (4)

where $\kappa = 2k \sin \frac{\theta}{2}$ and $k$ is the absolute value of the incident momentum. Substituting the potential we get

$$f(\theta) = -\frac{2m}{\hbar^2\kappa} \int_{0}^{\infty} ve^{-r^2/\alpha^2} \frac{1}{2\ell} (e^{i\kappa r} - e^{-i\kappa r}) r dr$$  \hspace{1cm} (5)

$$= i\frac{mv a^2}{\hbar^2\kappa} \int_{-\infty}^{\infty} e^{-x^2+i\alpha x} dx$$  \hspace{1cm} (6)

The integral can be evaluated in the following way

$$I(\alpha) = \int_{-\infty}^{\infty} e^{-x^2+i\alpha x} dx$$  \hspace{1cm} (8)

$$= -i \frac{\partial}{\partial \alpha} \int_{-\infty}^{\infty} e^{-x^2+i\alpha x} dx$$  \hspace{1cm} (9)

$$= -i \frac{\partial}{\partial \alpha} e^{-\frac{\alpha^2}{4}} \int_{-\infty}^{\infty} e^{-(x-\frac{\alpha}{2})^2} dx$$  \hspace{1cm} (10)

$$= i\frac{\alpha}{2} e^{-\frac{\alpha^2}{4}} \sqrt{\pi}$$  \hspace{1cm} (11)

Consequently

$$f(\theta) = -\frac{mv a^3}{2\hbar^2} e^{-\frac{(na)^2}{4}} \sqrt{\pi}$$  \hspace{1cm} (12)

$$\frac{d\sigma}{d\Omega} = \frac{m^2 v^2 a^6 \pi}{4\hbar^4} e^{-2(ka)^2 \sin^2 \frac{\theta}{2}}$$  \hspace{1cm} (13)

The incident energy is $E = k^2/2m$. The Born approximation works better for small scattered amplitude, i.e. for big momentum $k$ and the incident energy $E$. The ”classical” intuition is that particles with big velocity have small time to feel the influence of the potential, i.e. they scatter less than the particles with small velocity. And the Born approximation works better for the small scattering.
4. Exciting collision

(a) Energy conservation gives: 
\[ E_f = \frac{p_i^2}{2m} - \hbar \omega, \quad p_f = \sqrt{p_i^2 - 2\hbar m \omega}. \]

(b) Particle 1 moves with velocity \( v = p_i/m \). In the periodic box, it passes particle 2 every \( L/v = Lm/p_i \) seconds. The transition probability per collision \( P_{01} \) is obtained by multiplying the rate \( R_{01} \) (defined as the transition probability per unit time) by this time-interval.

(c) Inside the periodic box, the momentum is quantized via 
\[ p_f = \frac{2\pi n}{L}. \]
So 
\[ E_f = \frac{(2\pi)^2 \hbar^2 n^2}{2mL^2}, \quad n = \frac{L}{2\pi \hbar} \sqrt{2mE_f} \]

The density of states is
\[ \rho(E_f) = \frac{dn}{dE_f} = \frac{L}{4\pi \hbar} \sqrt{\frac{2m}{E_f}} = \frac{mL}{2\pi \hbar p_f} \quad (14) \]

(d) The normalized wavefunctions of particle 1 are: \( \langle x_1 | p \rangle = \frac{1}{\sqrt{L}} e^{\frac{i}{\hbar} p x_1} \). So we compute:
\[ \langle p_f | V_{int} | p_i \rangle = \frac{1}{L} \int dx_1 e^{\frac{i}{\hbar} (p_i - p_f)x_1} \delta(x_{12}) = \frac{\lambda}{L} e^{\frac{i}{\hbar} (p_i - p_f)x_2} \]

(e) Given this result we have:
\[ V_{if} = \frac{\lambda}{L} \langle \psi_1 | e^{\frac{i}{\hbar} (p_i - p_f)x_2} | \psi_0 \rangle \]

The harmonic oscillator wavefunctions are given in Griffiths, section 2.3.2:
\[ \psi_n(x) = \left( \frac{m \omega}{\pi \hbar} \right)^{\frac{1}{4}} H_n(\xi) e^{-\xi^2/2} \]
with 
\[ H_0(\xi) = 1, \quad H_1(\xi) = 2\xi, \quad \xi = \sqrt{\frac{\hbar}{m \omega}} x \]

So we get (using that \( \sqrt{\frac{m \omega}{\hbar}} \int dx = \int d\xi \))
\[ V_{if} = \frac{\lambda}{L \sqrt{\pi}} \int d\xi 2\xi e^{-\xi^2 + i\alpha \xi}, \quad \alpha = \frac{p_i - p_f}{\sqrt{m \hbar \omega}} \]
After a partial integration, this reduces to a simple gaussian integral:
\[ V_{if} = \frac{\lambda}{L} i \alpha e^{-\alpha^2 / 4} = \frac{\lambda}{L} i (p_i - p_f) \sqrt{\frac{m \hbar \omega}{4m \hbar \omega}} e^{-\left(\frac{p_i - p_f}{2\sqrt{m \hbar \omega}}\right)^2} \quad (15) \]
The final result for the transition probability $P_{01}$ is obtained by combining all above equations:

$$P_{01} = \frac{mL}{p_i} \cdot \frac{2\pi}{\hbar} \cdot |V_{rf}|^2 \cdot \frac{mL}{2\hbar p_f} = \lambda^2 \frac{m (p_i - p_f)^2}{\hbar^3 \omega p_ipf} e^{-\frac{(p_i - p_f)^2}{2\hbar\omega}}$$

(f) This is a useful exercise! Since $P_{01}$ is a probability, it must be dimensionless. Let’s first make sure that the exponent is dimensionless:

$$\frac{(p_i - p_f)^2}{2m} \cdot \frac{1}{\hbar\omega} = \text{energy} \cdot \frac{1}{\text{energy}}$$

Done. Now let’s use the same tactics for the prefactor. We can write it as:

$$\lambda^2 \cdot \frac{(p_i - p_f)^2}{\hbar^2} \cdot \frac{m}{p_ipf} \cdot \frac{1}{\hbar\omega}$$

Since $\int dx V_{int} = \lambda$ we see that $\lambda$ has dimension of energy $\times$ length. Now $(p_i - p_f)/\hbar$ has the dimension of a wavenumber, which is $1$/length. So the first two factors of the above expression have combined dimension of $(\text{energy})^2$. The last two factors each have dimension of $1$/energy. So combined the whole expression is indeed dimensionless.