Physics 103

Department of Physics
Princeton University

Precept Notes

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Harmonic Oscillator
& Exam Review

Part 1

Announcements

- This week, read chapter 14, sec 1-5.
  - There is no set due Monday.
- There is an exam next Tuesday.
- The exam covers chapters 12-14.5.
- Only basic aspects of harmonic motion will be covered: horizontal and vertical (combined with gravity)

Probable Exam Topics

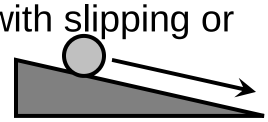
An outline of what we've done gives the main topics likely to be seen on an exam:

- 2d (fixed axis) rotational motion
- 3d rotational motion (vector $\boldsymbol{\tau} = d\mathbf{L}/dt$.)
- Orbits – circular and elliptical
  - Energy and angular momentum conservation with gravity
- Simple harmonic motion

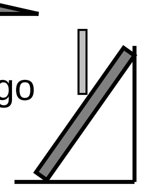
Relatively simple problems only – either horizontal or vertical (with gravity). Practice both.

2D Rotation Problems

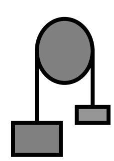
- Torque, Moment of Inertia calculations
  (Practice doing the integral, especially in 1d.)
- Rolling problems, possibly with slipping or inclined planes, loops, etc.



- Statics problems
  (slippery ladder, hanging sign, …)

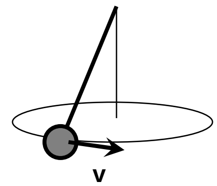


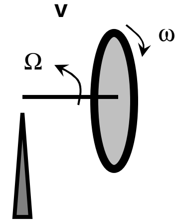
- Angular momentum conservation (merry go round with people jumping on it, …)
- Rigid body dynamics: F = ma and $\tau = I\alpha$
  (masses with massive pulley, …)



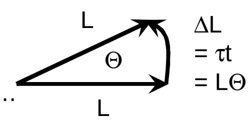
3D Rotation Problems

- These come in 2 flavors, mostly – the rotating mass problem (#2 last Friday) and the precessing pendulum (#3 last Friday, or the bicycle problem in the homework).
- Make sure you can do both of these, because there is a high probability you will need to do one or the other. Either way, practice the right-hand rule and remember what happens to a vector that is pushed perpendicular to itself.





practice understanding this picture, even if you must memorize it…



Conservation of Angular Momentum

- Angular momentum conservation with the general definition **L** = **r**x**p** can also come up.
- We've seen this mostly in orbit problems, where it is very useful for non-circular orbits (elliptical or non-periodic orbits).
- It also appears in rigid body collisions, where both **L** and **p** are conserved. We may do one of these next Monday if time permits.

Bicycle Problem

The bicycle problem confused a lot of people. The idea was you have a weight hanging off, say, the left side of the bike, and steer the wheel to the left so that the torque you put on the wheel to turn it (actually, its reaction torque in the bike) cancels the torque that topples the bike.

In real life, the gyroscopic effect of a bicycle wheel turns out to be relatively minor compared to the effect of just turning in the direction the bicycle is leaning.

Consider the following example, where there is no wheel at all, and so minimal angular momentum in the problem (which we'll actually neglect completely).

We'll still see that just leaning into a curve creates stability, or conversely, if you start falling to the left, lean that way, and you can have a stable path.

Figure Skater

A figure skater makes a circular path of radius R at speed v. To keep from falling, she leans inward. What angle should she lean into the curve?

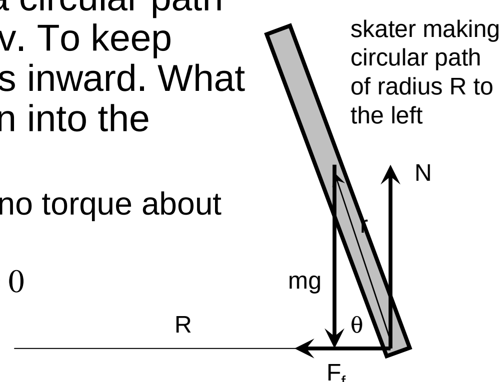
Not tipping means there is no torque about her CM.

$\tau_{cm} = F_f\, r \cos\theta - N\, r \sin\theta = 0$

$N = mg$, $F_f = mv^2/R$.

$v^2/R = g \tan\theta$.

$\theta = \mathrm{atan}\,(v^2/Rg)$.

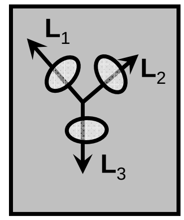


Torques do not balance about a point other than her CM! They don't balance around her foot.

Fine print: Technically, there is a small torque about the CM which rotates the tilted skater slowly about her vertical axis when she makes one full circle. We've neglected that, which is ok if R >> r.

Note on Vector Nature of L

Angular momentum really adds as a vector. I demonstrated it on a stool, but that was just a 1d example. You could attach several gyroscopes to a board and find that the angular momentum vectors add to a total conserved angular momentum vector $\mathbf{L}_1 + \mathbf{L}_2 + \mathbf{L}_3$. If you adjust the speeds so that $\mathbf{L}_1 + \mathbf{L}_2 + \mathbf{L}_3 = 0$, you lose gyroscopic stability totally.



The **L**'s add in 3d also, but you can't adjust three to have **L** = 0 if they are all perpendicular.

Orbits

Kepler's 2nd Law is really just a statement of angular momentum conservation, interpreted geometrically.

Kepler's 3rd Law is a result of E and L conservation. Use the form $\mathbf{L} = m\mathbf{r}\times\mathbf{v}$.

Be sure to practice the circular orbits too: they are simpler. You can use $mv^2/r = GMm/r^2$. This doesn't apply to other orbits.

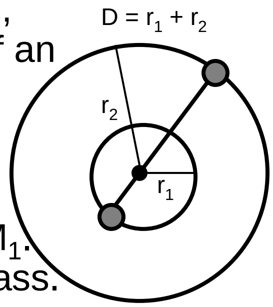
Derive any relations such as $E = -GMm/2a$ if you use them. (a = semimajor axis). The derivation is easy for a circular orbit.

Binary System

- You had a problem with a binary system.
- Then you still use $F = GM_1M_2/D^2$
- If the planets have the same period, then they rotate at opposite ends of an axis at a common angular rate ω.

  $M_1r_1\omega^2 = GM_1M_2/(r_1 + r_2)^2$,

  $M_2r_2\omega^2 = GM_1M_2/(r_1 + r_2)^2$.
- Note that dividing gives $r_1/r_2 = M_2/M_1$. They rotate about their center of mass.



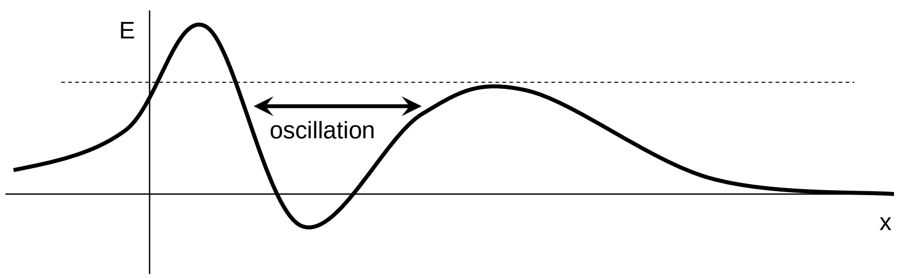
Oscillatory Motion

Oscillation happens about a minimum of the potential energy. Total energy is conserved.

For what range of motion can oscillation occur?

Where is the motion fastest?

Where are the turning points?



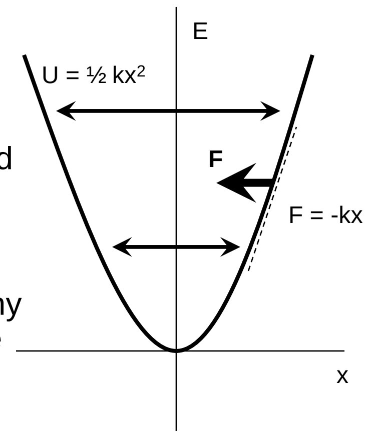
Simple Harmonic Motion

If the period doesn't depend on the amplitude, the motion is called **simple harmonic**.

Note: a bell oscillates when struck. The oscillations produce the sound we hear, and the frequency of oscillation determines the pitch.

If the frequency doesn't depend on the amplitude, hitting the bell at any strength should produce the same note. This is why we call it "harmonic".

A quadratic P.E. or linear restoring force produces SHM.



Small Oscillations

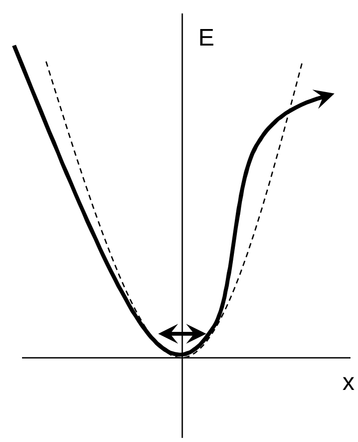
Even if the PE is not perfectly parabolic, but is smooth, F = 0 at the minimum, and is approximately F = -kx near the minimum.

Minimum: dU/dx = 0.

Near minimum,

$F = -x\,(dF/dx)_{x=0} = -kx$

$k = dF/dx = d^2U/dx^2$.



This makes harmonic oscillators ubiquitous throughout physics!

Simple Harmonic Motion

Oscillation happens about a minimum of the potential energy. $U + K = E$.

Fastest motion at the equilibrium position.

Motion of a mass on a spring.

Let $x = 0$ be the equilibrium position.

$F = -kx$

$a = -(k/m)\,x$

$U = ½\,kx^2$

$½\,mv^2 + ½\,kx^2 = E$

How to find the motion $x(t)$? There is a very useful trick…

Projected Circular Motion

Remember Uniform Circular Motion:

$a = v^2/r = r\omega^2$ inward.

$\mathbf{a} = -\omega^2\,\mathbf{r}$ as a vector.

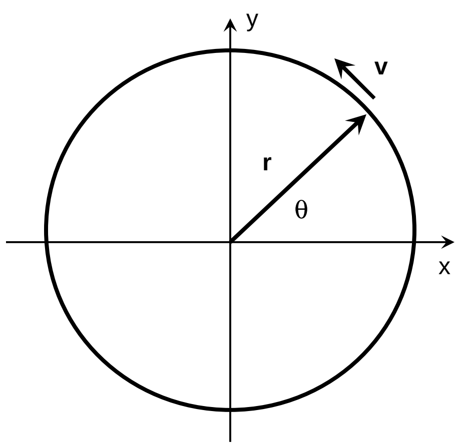
force:

$\mathbf{F} = -m\omega^2\,\mathbf{r}$.

$F_x = -kx$, $F_y = -ky$

with $k = m\omega^2$.

The amplitude is $A = r$.



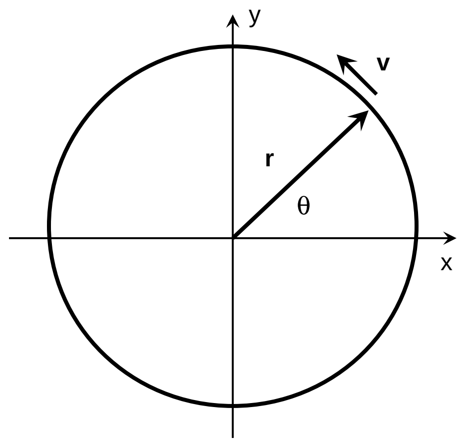
Projected Circular Motion

acceleration:
$\mathbf{a} = -\omega^2 \mathbf{r}$.
$a_x = -(k/m)x$
with $k = m\omega^2$.

position:
$x = r \cos\theta \qquad \theta = \theta_0 + \omega t$

velocity:
$v_x = -v \sin\theta = -vy/r$
with $y = (r^2 - x^2)^{1/2}$

period:
$T = 2\pi/\omega = 2\pi\,(m/k)^{1/2}$
(independent of r)



Equation for SHM

If I push a block of mass m on a spring with spring constant k, giving it initial speed v_0 at position x_0, what is the equation of motion?

Remember: $x = A \cos\theta$ where the angle is $\theta = \theta_0 + \omega t$ with frequency $\omega = (k/m)^{1/2}$.

A, θ_0 must be determined from the initial conditions.

Equation for SHM

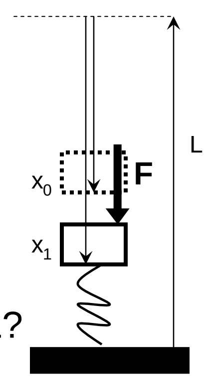
$x_0 = A \cos \theta_0$.
$v = dx/dt = -A (\sin \theta)\, d\theta/dt = -A\omega \sin \theta$
$v_0 = -A\omega \sin\theta_0$.

$A = A (\sin^2\theta_0 + \cos^2\theta_0) = (v_0/\omega)^2 + x_0^2$.
$\tan \theta_0 = -v_0/\omega x_0$.

- Note the similarity to finding the magnitude and angle of a vector. As usual – it's not the result that's significant, but the way we got it.
- Also note that you don't really need calculus to find v(t), since you can use the circular motion:

  The facts that $v = r\omega$ and **v** is perpendicular to **r** imply that $v = -r\omega \sin \theta$ when $x = r \cos \theta$. (Replace r by A.)

Vertical Spring



A mass m is set on a vertical spring of uncompressed length L. It sinks down a distance x_0 below L. What is k?

$$mg = kx_0$$

What *additional* force F does it take to push the mass a distance x_1 below L?

$$F + mg = kx_1, \quad mg = kx_0$$

imply Hooke's Law about x_0:

$$F = k(x_1 - x_0)$$

This gives SHM about x_0 with $\omega^2 = k/m$.

Vertical Spring

Note: $F = k(x_1 - x_0)$ is Hooke's law for equilibrium position x_0.

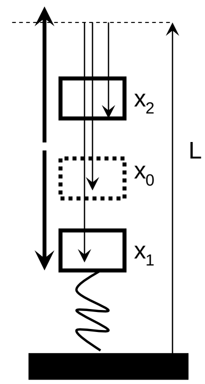
This means that if the mass is released, and remains attached to the spring, it will oscillate up and down about x_0.

The amplitude is $x_1 - x_0 = F/k = x_0F/mg$.

The highest point reached in the oscillation is a distance

$x_2 = x_0 - (x_1 - x_0) = 2x_0 - x_1$ below the unstretched position.

This distance can also be written as $x_2 = x_0(1 - F/mg)$.

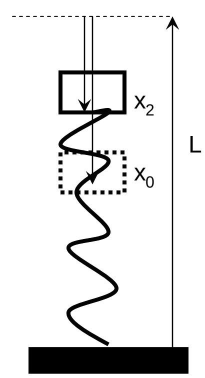


Vertical Spring

High point: $x_2 = x_0(1 - F/mg)$.

If $x_2 < 0$, it rises above the unstretched length of the spring, and above that point, the spring is pulling down.

If the object is not attached to the spring, it will launch at this point. (The Normal force is zero at height L, since it is the spring's equilibrium position.)



Vertical Spring

Where is the object moving fastest?

At its equilibrium point, $L – x_0$.

If the mass stays attached to the spring, what is its oscillation frequency?

$\omega = (k/m)^{1/2}$, $k = mg/x_0$

$\omega = (g/x_0)^{1/2}$

Vertical Spring

Where is the acceleration the greatest?

At the top and bottom (x_2, x_1), the net force and acceleration are maximum:

$a = kA/m = k(x_1 – x_0)/m$ or alternately,

$= \omega^2 A = (x_1 – x_0)\omega^2$.

The forces at the top and bottom are the same:

Bottom:

$F_1 = kx_1 – mg = k(x_1 – x_0)$.

Top:

$F_2 = kx_2 – mg = mg + k(2x_0 – x_1) = k(x_0 – x_1)$.