Physics 103

Department of Physics
Princeton University

Precept Notes

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November 19, 2007

Rigid Body Motion and Statics

Part 3

3D Rotations & Orbits

Announcements

- There are no labs this week. There is an exam next Tuesday. Happy Thanksgiving!
- This week, read chapter 14, sec 1-5.
  - There may not be a set due Monday – Check WebAssign for announcements.
- This is on simple harmonic motion.
  - This is a type of oscillating motion in which the amplitude doesn't depend on the period.
  - A mass attached to the end of a spring oscillates this way.

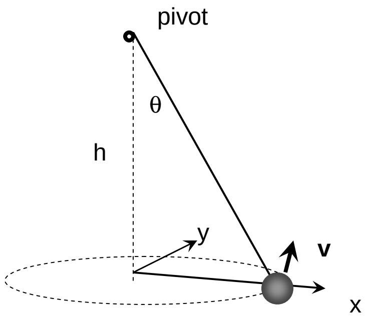
Mass on String

A ball circles on the end of a string. The speed is v.

Is the angular momentum about the pivot constant?

What is **L** = **r**x**p** when the ball is on the +x axis?

Tak θ = 30^o.



Mass on String

Method 1 (geometry):

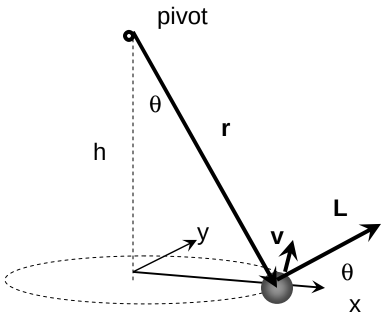
L = rmv.
L_x = rmv cos θ = hmv
L_y = 0
L_z = rmv sin θ = mv h tanθ = hmv/$3^{1/2}$

Method 2 (components):

r_x = h tan θ = h/$3^{1/2}$.
r_y = 0, r_z = – h

p_x = 0, p_y = mv, p_z = 0

$L_x = r_y p_z - r_z p_y$ = hmv
$L_y = r_z p_x - r_x p_z$ = 0
$L_z = r_x p_y - r_y p_x$ = hmv/$3^{1/2}$



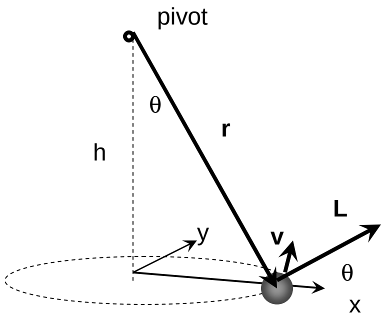
Mass on String

What happens to **L** as the ball rotates? Where does it point when the ball is on the +y axis?

L_y = hmv,

L_z stays the same.

The “transverse” component of **L** is L_t = hmv, directed radially outward in the xy plane.



Mass on String

Note that only

$L_z = hmv/3^{1/2} = mvd$

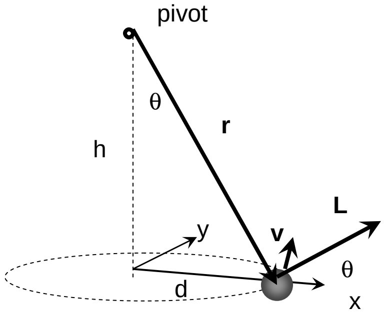
remains constant.

You could have calculated this component in 2d.

In general, if something is rotating about an axis, and you call that the z axis, then $L_z = I\omega$ with I the moment of inertia about the rotation axis.

$I = md^2$, $\omega = v/d$, $L_z = I\omega = mvd$.

You can’t use $L = I\omega$ in the other directions.



Mass on String

What is the torque vector about the pivot? Which components are changing?

$\tau = d\mathbf{L}/dt$. The torque rotates around the circle, parallel to **v**.

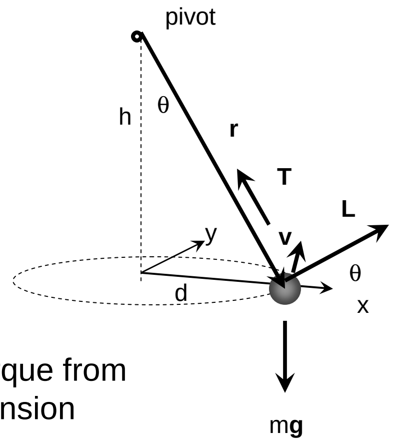
Direct calculation:

$\boldsymbol{\tau} = \mathbf{r} \times m\mathbf{g}$

no torque from the tension

$\tau_x = mgr \sin\theta$

$= mgd = mgh/3^{1/2}$



Note: $\tau_z = 0$, so L_z is conserved.

Mass on String

$\tau = mgd$

$dL_t/dt = L_t\omega$. (like $ds/dt = R\omega$)

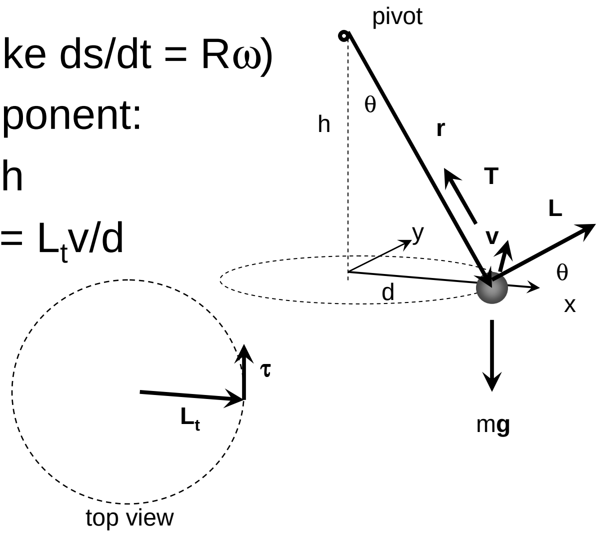
transverse component:

$L_t = mvh$

$\tau = dL_t/dt = L_t\omega = L_t v/d$

$mgd = mv^2h/d$

So $v^2 = gd^2/h$



Mass on String

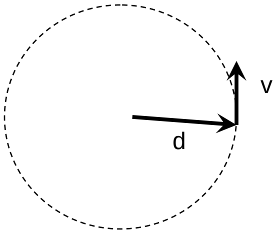
How else could we find that $v^2 = gd^2/h$?

Centripetal force:

$mv^2/d = T \sin \theta = T\, d/r$

$T \cos \theta = T\, h/r = mg.$

$v^2/d = mgd/h.$

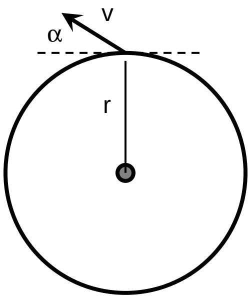


Note: situations with a changing **L** are said to be in rotational imbalance.

Change of Orbit

If I am in a circular orbit at radius r about the earth and suddenly change my direction by angle α without changing my speed, what are the energy and angular momentum of the new orbit?

What is the closest and furthest approach to the earth (perigee and apogee) for the new orbit?



You can change your direction without changing your speed by firing a rocket perpendicular to your direction of motion. This does no work.

Orbit Problems

For a circular orbit, you can always use the fact that the gravitational acceleration is the same as the centripetal acceleration:

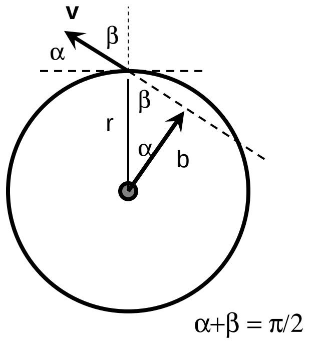
$$v^2/r = GMm/r.$$

For any other orbit, you cannot use this relation. Just having **v** perpendicular to **r** is not enough to use this relation: you **can't** use it at the ends of the semimajor axis of an elliptical orbit.*

For non-circular orbits, use conservation of total energy and angular momentum as the starting point.

*We'll see in the next problem that it does hold at the ends of the semi**minor** axis, but this is not obvious. The circular orbit with the same semimajor axis intersects the ellipse at these two points.

Change of Orbit



Energy is the same.

$E = \frac{1}{2} mv^2 - GMm/r$

with $mv^2/r = GMm/r^2$

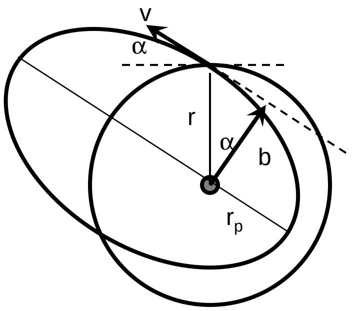
$v^2 = GM/r$

$E = -GMm/2r$

$L = mvb = mvr \cos \alpha$

$\quad = m(GMr)^{1/2} \cos \alpha$

Change of Orbit



$L = mvb = mvr \cos\alpha$

$\quad = m(GMr)^{1/2} \cos\alpha$

Closest approach:

$L = mr_pv_p$.

$E = ½\, mv_p^2 – GMm/r_p$

$\quad = – GMm/2r$

Use $v_p^2 = GMr/r_p^2 \cos^2\alpha$

$(GMr/2r_p^2) \cos^2\alpha – GMm/r_p + GMm/2r = 0$.

Change of Orbit

Multiply by 2r/GMm, obtaining

$(r^2/r_p^2) \cos^2\alpha – 2r/r_p + 1 = 0$.

Solve the quadratic equation to get

$r/r_p = [2 \pm (4 – 4\cos^2\alpha)^{1/2}] / (2 \cos^2\alpha)$

$\quad = (1 \pm \sin\alpha)/\cos^2\alpha$. (We need the bigger solution.)

$r_p/r = (1 – \sin^2\alpha)/(1 \pm \sin\alpha) = 1 – \sin\alpha$.

The other solution gives the furthest approach $r_a/r = 1 + \sin\alpha$.

Note that $r_a + r_p = 2r$, $r_a – r_p = 2\sin\alpha$.

Change of Orbit

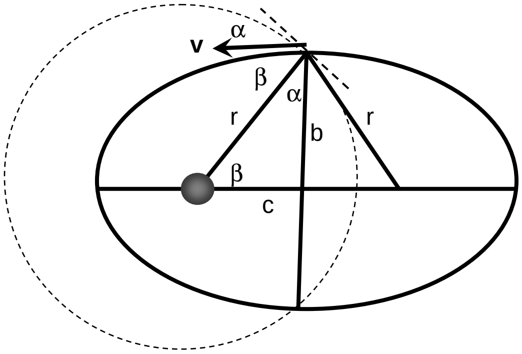
Since the orbit is an ellipse with major axis 2r, r = a and the transfer must have occurred at the end of a semi-minor axis.

b = a cos α.

c = a sin α.

The eccentricity of the orbit is

$e = c/a = \sin \alpha$.



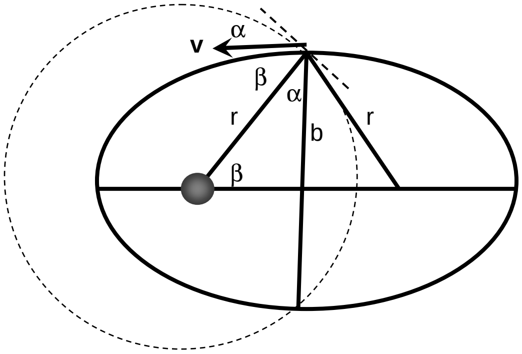
α and β are complementary angles.

Change of Orbit

At the ends of the minor axis, a satellite is going as fast as a satellite in a circular orbit with the same major axis,

$v_{circ} = (GM/a)^{1/2}$.

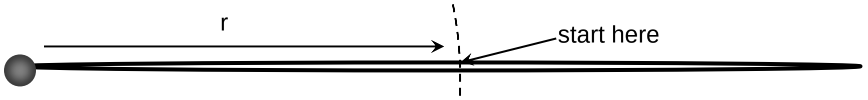
$L = mbv_{circ}$



α and β are complementary angles.

Limiting Cases

b = a gives a circular orbit. What about b = 0?



The orbit is squashed to a line. You have the object bounces elastically off the earth without being destroyed, but if it could, it would bounce out twice as far as the point where it left the circular orbit, as if it had been dropped from distance 2r. Notice that the foci are at the ends of the orbit.

Unbound Orbits

Unbound orbits have E ≥ 0.

If E = 0, the orbit is parabolic.

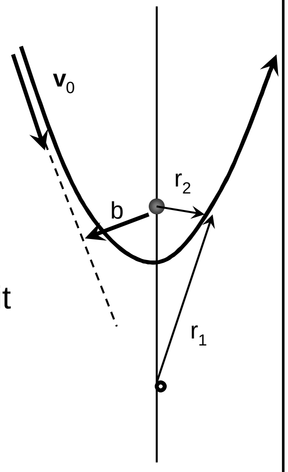
If E > 0, the orbit is hyperbolic.

Parabolic orbits correspond to objects at rest at infinity.

The angular momentum of a hyperbolic orbit can be determined when the object is very far from the sun and moving in a straight line.

$L = mbv_0$, where b is the distance it would miss the sun by if it kept going straight.

b is called the “impact parameter”.



$r_1 - r_2$ = constant