Physics 103

Department of Physics
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Precept Notes

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Wave Motion

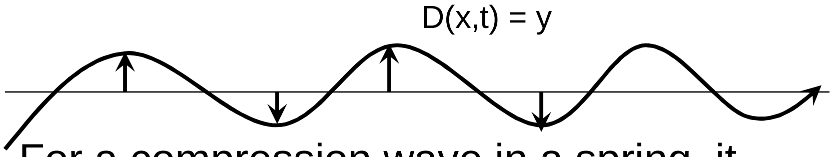
Part 1

Announcements

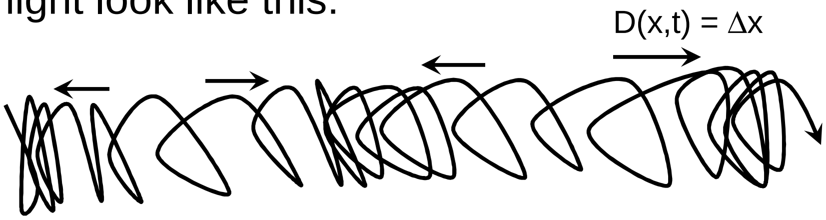
- We are now on Knight Chapters 20 – 21.
- The topic is wave motion.
- Today we will mostly discuss topics from Chapter 20, and on Monday, mostly topics from Chapter 21.
- Problem sets have been posted through the end of the semester.
- I strongly recommend trying the interference problem in Monday’s set (antennas) before Friday.

Waves

- A snapshot of a wave on a string at fixed time may look like this:



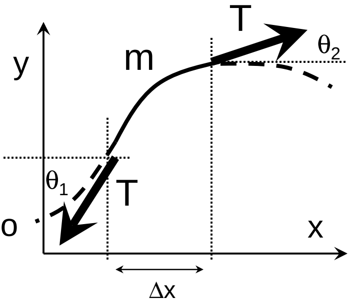
- For a compression wave in a spring, it might look like this:



Wave Equation

- The motion of the material is back and forth, but the vibration doesn’t stay there: it travels through the medium.
- It is the shape of the wave that travels – the individual pieces of material just vibrate back and forth where they are.
- Applying Newton’s Law to a little piece of vibrating string gives the **wave equation**.

Wave Equation



Consider a little piece of string of linear mass density μ.

The picture is exaggerated: normally the displacements y are small.

$m = \mu dx$ moves up and down due to the net force in the y direction:

$T(\sin\theta_2 - \sin\theta_1) = ma$.

Approximate

$\sin\theta \approx \tan\theta = dy/dx$ and take Δx small. Then

$$\sin\theta_2 - \sin\theta_1 = (dy/dx)_2 - (dy/dx)_1 \approx (d^2y/dx^2)\,\Delta x.$$

$$\boxed{Td^2y/dx^2 = \mu a.}$$

Wave Equation

Then $d^2y/d^2t = (T/m)\, d^2y/d^2x$.

- The derivatives are taken with the other variable held fixed, ie, the variables x and t are mathematically independent.
- For a function of several variables, such derivatives are called partial derivatives, and usually written with curly d's, ∂ .

$$\partial^2y/\partial^2t = (T/m)\, \partial^2y/\partial^2x.$$

This is a second order linear partial differential equation – but one of the easiest to solve!

Wave Equation

Write the positive constant $T/m = c^2$, so that c has dimensions m/s.

A solution can be found to

$$\partial^2y/\partial^2t = c^2\ \partial^2y/\partial^2x.$$

if we solve $\partial y/\partial t = \pm\, c\ \partial y/\partial x$.

This has many solutions.

Take $y(x,t) = D(x \pm ct)$.

Then $\partial y/\partial x = D'(x \pm ct)$ and $\partial y/\partial t = \pm\, c\ D'(x \pm ct)$.

This works for any function f at all, so there really are infinitely many solutions to the wave equation. In words, they are waves or pulses that hold their shape as they travel either way on the string at speed c. Such waves are called **nondispersive**, and are found whenever c is really a constant. (It sometimes depends on other things.)

Wave Equation

The function $D(x – ct)$

is shown at time 0. How will it look at time $t > 0$?

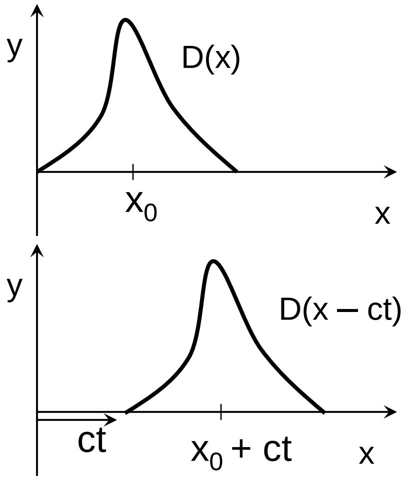
The point $x = ct$ will now have displacement $D(x)$.

The point $x_0 + ct$ will now have displacement

$D(x_0)$.

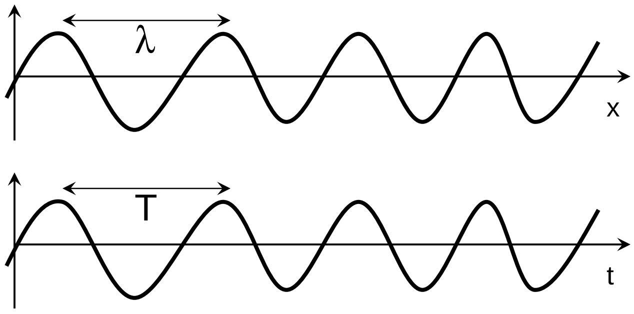
The pulse moves right at speed c.

Similarly, $D(x + ct)$ describes a left-moving pulse of the same shape.



Periodic Waves

- Periodic solutions have a wavelength λ (the period in space) and frequency ($1/T$).



Sine Waves

The simplest periodic waves are the sine waves,

$$y(x,t) = A \sin(2\pi(x/\lambda - t/T) + \phi).$$

It is convenient to use $k = 2\pi/\lambda$, $w = 2\pi/T$ to

write this as $y(x,t) = A \sin(kx - \omega t + \phi)$.

Through Fourier analysis, any wave can be represented as a sum of sine waves. So it is really enough to understand sine waves.

Example

Which of the following waves move right?

(a) $y = A \sin(x - t)$

(b) $y = A \sin(x + t)$

(c) $y = A \sin(-x - t)$

(d) $y = A \sin(t - x)$

(e) $y = A \sin(x - t + \pi/2)$

(f) $y = A \sin(t - x - \pi/2)$.

Example

What are the period, wavelength and wave velocity of each?

(a) $y = A \sin(x - t)$

(b) $y = A \sin(x + t)$

(c) $y = A \sin(-x - t)$

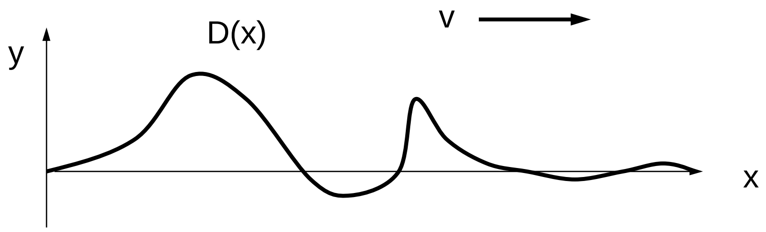
(d) $y = A \sin(t - x)$

(e) $y = A \sin(x - t + \pi/2)$

(f) $y = A \sin(t - x - \pi/2)$.

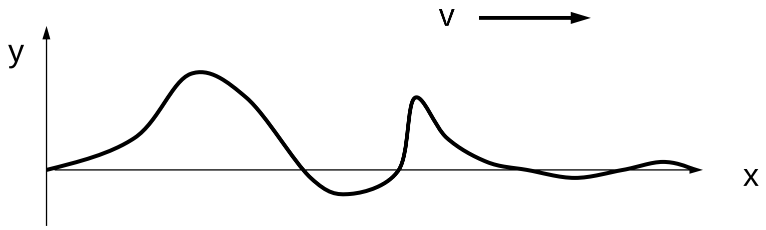
Wave Dynamics Exercise

- Below is a snapshot at time t = 0 of a wave on a rope moving to the right at velocity v. Where is the rope moving the fastest? Where is the velocity upward?



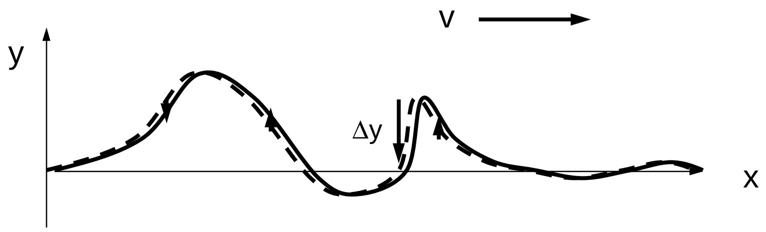
Wave Dynamics Exercise

- The entire waveform is moving to the right at a constant speed, but the rope is moving up and down with a changing speed as the wave goes by.



Wave Dynamics Exercise

- The biggest movement Δy occurs where the rope's waveform is the steepest.



Wave Dynamics Exercise

- Therefore, the rope's speed $v_y = \Delta y/\Delta t$ is greatest where the slope of the rope is greatest.
- For a general time, $y = D(x - c\,t)$ where $D(x)$ is the function shown at $t = 0$.
- The speed of the rope at position x is

$$v_y = \partial D(x - c\,t)/\partial t = - c\,D'(x - ct)\ .$$

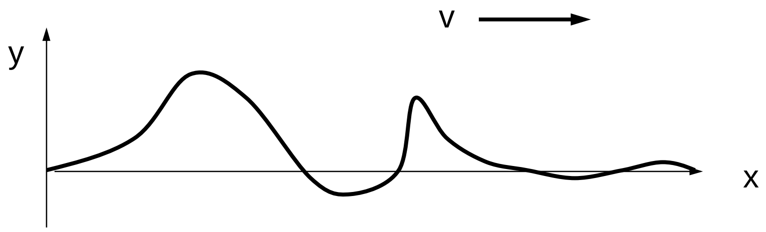
In the snapshot shown at $t = 0$,

$$v_y = - cD'(x).$$

The velocity is upward when the slope of $D(x)$ is downward.

Wave Dynamics Exercise

- Where is the transverse acceleration of the rope greatest?

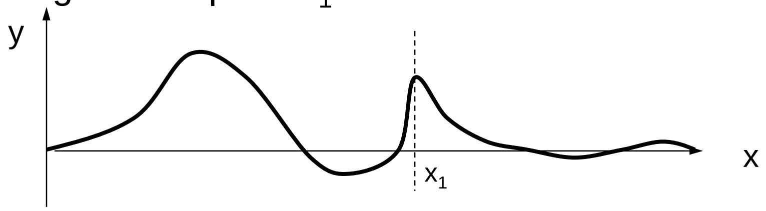


Wave Dynamics Exercise

The transverse acceleration of the rope is

$$a_y = \partial v_y/\partial t = - c\, \partial D'(x - ct)/\partial t$$
$$= c^2 D''(x - ct),$$

which is greatest where the curvature is greatest: point x_1.



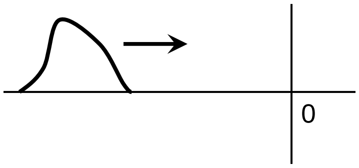
Boundary Conditions

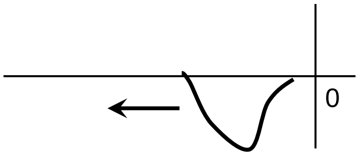
What if a wave comes in from the – x axis and hits a point where the string is tied down at $x = 0$? Write a solution to the wave equation in this case.

We can form a linear combination of a left and right moving wave so that $y(0) = 0$ at all times:

$$y = D(x - vt) - D(-x - vt).$$

This has the desired properties. $D(x - vt)$ is the wave traveling to the right. $D(-x - vt)$ is a mirror reflection of $D(x)$ traveling to the left. The minus sign means the reflected wave is inverted. Only values $x < 0$ count.





Boundary Conditions

A sine wave solution for reflecting off a fixed point at $x = 0$ would be

$$y = \sin(kx - \omega t) - \sin(-kx - \omega t)$$
$$= \sin(kx - \omega t) + \sin(kx + \omega t)$$
$$= \sin(kx)\cos(\omega t) - \cos(kx)\sin(\omega t)$$
$$+ \sin(kx)\cos(\omega t) + \cos(kx)\sin(\omega t)$$
$$= 2\sin(kx)\cos(\omega t).$$

This is a solution to the wave equation formed by adding two other solutions, though it doesn't look like it has the same form as our other solutions $y = D(x \pm ct)$.