Physics 103

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Precept Notes

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December 3, 2007

Nearly Harmonic Oscillators
Damped and Forced Oscillators
Part 2

Solving Newton's Equation

- Newton's Law normally has the form $F(t) = m\, d^2x/dt^2$. This is a second order differential equation
- In the case of a conservative force, we can construct $U(x)$ and convert this to a first order equation: $\frac{1}{2} mv^2 = E - U(x)$, so

$$dx/dt = (2/m)(E - U(x))^{1/2}$$

This has the form $dx/dt = f(x)$, and can be solved by integrating $dt/dx = 1/f(x)$ to get

$$t(x) = \int dx/f(x).$$

We did this last time. Now we want to consider the second order differential equation directly.

Linear Differential Equations

Spring equation F = ma can be written

$\ddot{x} = -(k/m)x.$ (Newton used dots for time derivatives.)

This is second order. It is nice to work with one order at a time. Suppose $\ddot{x} = +\omega^2 x$ instead.

It could be solved in steps if $\dot{x} = \pm\omega x$.

What function is proportional to its own derivative? The exponential

$$x = e^{\pm\omega t}$$

solves the equation. In general, you can form a combination $x = Ae^{\omega t} + B\,e^{-\omega t}$ for any A, B. These are all the solutions.

Linear Differential Equations

But we have the wrong sign to take a square root:

$\ddot{x} = -\omega^2 x.$

Imaginary numbers fix the problem.

It could be solved in steps if $\dot{x} = \pm i\omega x$.

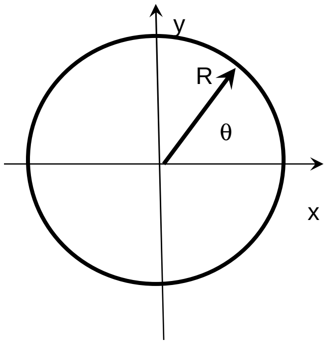
What function is proportional to its own derivative? The exponential

$$x = e^{\pm i\omega t}$$

solves the equation. In general, you can form a combination $x = Ae^{i\omega t} + B\,e^{-i\omega t}$ for any A, B.

Interpretation of Complex Solution

Complex numbers are 2-dimensional:
  $z = x + iy$ is equivalent to a 2-component vector.



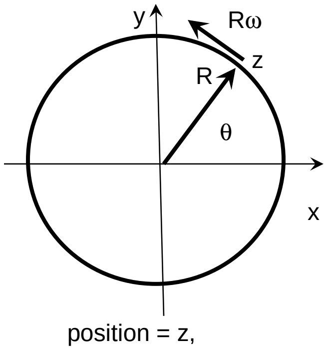
$e^{i\theta}$ is a unit vector at angle θ:

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

$z = Re^{i\omega t}$ is a point rotating about the origin at radius R with angular velocity ω.

Interpretation of Complex Solution

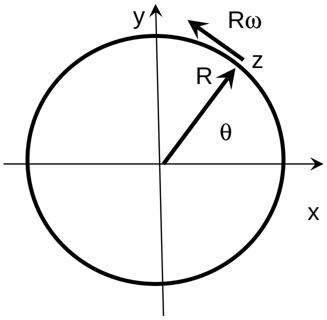
The velocity vector
is $dz/dt = i\omega Re^{i\omega t}$
  $= R\omega(\sin\theta - i\cos\theta)$.
Multiplying any complex number z by i rotates it 90^o counter-clockwise.



position = z,
velocity = $i\omega z$.

Interpretation of Complex Solution

The complex solution to $\ddot{z} = -\omega^2 z$ really describes the rotational analog of SHM. To get SHM, project onto the x axis by taking the real part.



$x = \text{Re } z$

with $z = Ae^{i\omega t} + Be^{-i\omega t}$.

Both rotation directions describe the same SHM, so only one is actually need, A or B. The coefficients A and B are complex in general.

Interpretation of Complex Solution

Any complex number can be written either as $z = x + iy$ or $z = re^{i\theta}$. The relation is familiar from 2d vectors:

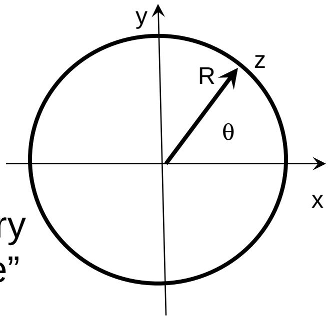
$r = x^2 + y^2$,

$\tan \theta = y/x$.

The magnitude of z is defined to be r: $|z| = r$.

Changing the sign of the imaginary part gives the "complex conjugate" $z^* = x - iy$. Note that

$|z|^2 = zz^*$, $\text{Re } z = \frac{1}{2}(z + z^*)$, $\text{Im } z = (z - z^*)/2i$.



Complex Arithmetic

Complex arithmetic works like regular arithmetic (commutative, distributive, associative laws…) with $i^2 = -1$.

When dividing, it is convenient to use the fact that $zz^* = |z|^2$, so that $w/z = wz^*/|z|$.

That way, you only have multiplication to consider among the complex numbers.

General Solution to SHM

Two complex parameters A and B depend on 4 real parameters, but only two parameters are actually needed to completely describe a solution, the initial position and velocity. Projecting onto the x axis makes it unnecessary to give the direction of the rotational motion: the solutions with $\pm\omega$ are equivalent for describing motion along the x axis. Only one is needed.

Taking $B = 0$ and calculating the real part gives

$$x = A_0 \cos(\omega t + \phi),$$

the general solution for SHM, with $A = A_0 e^{i\phi}$.

Viscosity

Viscosity is described by $F = -bv$. (b = kg/s)

Then $a = -(b/m)v = -\gamma v$. The equation

$$dv/dt = -\gamma v$$

can be solved via an exponential function with arbitrary coefficient, $v = v_0\, e^{-\gamma t}$. If an object starts out at speed v_0, this gives the speed at later times. The speed is reduced by a factor $1/e$ in time $1/\gamma = m/b$ (seconds).

Viscosity

If an object is dropped in a liquid with viscosity bv, then it falls according to

$$dv/dt = g - \gamma v.$$

It reaches terminal velocity when $g = \gamma v$.

Thus, $v_t = g/\gamma$. Or, $dv/dt = g(1 - v/v_t)$.

$v = vt(1 - e^{-\gamma t})$ gives $dv/dt = ge^{-\gamma t}$ by substitution on the right. On the left, it gives $dv/dt = \gamma v_t e^{-\gamma t}$. These are equal since $\gamma v_t = g$.

Viscosity

If an object is dropped in a liquid with viscosity bv, then it falls according to

$$dv/dt = g - \gamma v.$$

It reaches terminal velocity when $g = \gamma v$.

The initial acceleration is g, but eventually the acceleration is canceled and the falling object attains a steady state of falling at constant velocity $v_t = g/\gamma$. This can be used to measure γ, in practice.

Damped Harmonic Oscillator

If a harmonic oscillator is vibrating in a viscous medium, $F = -kx - bv = ma$.

Then $a + \gamma v + \omega_0^2 x = 0.$

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0.$$

The solution is by guessing or experience. Since it worked so well for the harmonic oscillator, we'll **try** a solution:

$$x = Ae^{i\omega t}$$

with ω to be determined. A will be a free parameter. If we can get two different solutions of this form, we'll have the general solution by adding them.

Damped Harmonic Oscillator

Substitute $x = Ae^{i\omega t}$ into

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0.$$

and find an equation

$$-\omega^2 x + i\gamma\omega x + \omega_0^2 x = 0.$$

The solution x = 0 is not interesting. This implies that there are two choices of ω in general that give a solution:

$$\omega = -\gamma/2 \pm [(\gamma/2)^2 - \omega_0^2]^{1/2}$$

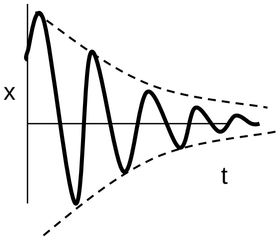
Damped Harmonic Oscillator

$\omega = -\gamma/2 \pm [(\gamma/2)^2 - \omega_0^2]^{1/2}$

will give an **oscillating** solution if it has an imaginary part. That happens if $\omega_0 > \gamma/2$, i.e., $k > b^2/4m$.

Let $\omega = -\gamma/2 \pm i\omega_1$. Then

$$z = A\, e^{-\gamma/2}\, e^{i\omega_1 t} + B\, e^{-\gamma/2}\, e^{-i\omega_1 t}.$$



You can't tell which whether the circular motion is clockwise or counterclockwise from its x projection, so you can set $A = A_0 e^{i\phi}$, $B = 0$.The real part is then

$$x = A_0 e^{-\gamma/2} \cos(\omega_1 t + \phi)$$

This solution is called "underdamped".

Damped Harmonic Oscillator

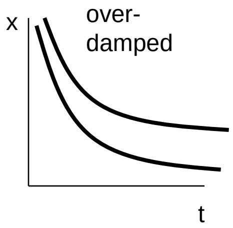
If $\omega_0 < \gamma/2$, then

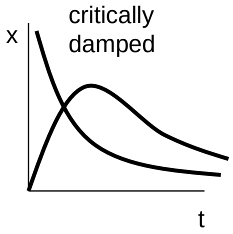
$\omega = -\gamma/2 \pm [(\gamma/2)^2 - \omega_0^2]^{1/2}$ is real and gives $\omega = -\gamma_{1,2}/2$ for the two solutions.

$x = A\,e^{-\gamma_1 t/2} + B\,e^{-\gamma_2 t/2}$ is real and decays exponentially.

In the critical case, $\omega = -\gamma/2$ is the only solution. There is a special solution,

$$x = (A + B\gamma t)e^{-\gamma t/2}.$$





Special cases are one of the annoying aspects of differential equations.
If you only get one solution and need another, try multiplying it by something linear.

Energy in Damped Oscillation

In an undamped harmonic oscillator, energy is conserved, ½ kA^2. With underdamping, the amplitude decreases exponentially, and so does the energy:

$$E = \tfrac{1}{2} kx^2 + \tfrac{1}{2} mv^2$$
$$= \tfrac{1}{2} Ae^{-\gamma t}\,[k \cos^2(\omega t+\phi) + m\omega^2 \sin^2(\omega t+\phi)]$$
$$= E_0 e^{-\gamma t}\,[\cos^2(\omega t+\phi) + (\omega/\omega_0)^2 \sin^2(\omega t+\phi)].$$

If $\gamma << \omega_0$, then $\omega \approx \omega_0$ and using $\sin^2\theta + \cos^2\theta = 1$ gives

$$E \approx E_0 e^{-\gamma t}.$$

$E = \tfrac{1}{2} E_0$ when $e^{-\gamma t} = \tfrac{1}{2}$, so that $t = \ln 2/\gamma$. This is called the "half life" of the energy in the damped oscillator.

Forced Harmonic Oscillator

An oscillator can be “pumped” with frequency ω. If ω is close to the natural frequency of oscillation, the amplitude can increase dramatically. Apply an oscillating force $F = f_0 e^{i\omega t}$. (The actual force is $f_0 \cos \omega t$.) Take a trial solution $z = A e^{i\omega t}$. All the terms in the equation

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = a_0 e^{i\omega t},$$

with $a_0 = f_0/m$, then have the same functional form with different constant coefficients. Solving the equation is reduced to algebra much as before.

Forced Harmonic Oscillator

The amplitude is no longer a free parameter, but is related to the driving acceleration by

$$a_0 = A(\omega_0^2 - \omega^2 + i\gamma\omega).$$

Let $w = \omega_0^2 - \omega^2 + i\gamma\omega$. Then

$$A = a_0/w = a_0 w^*/|w|^2 = a_0(w^*/|w|)/|w|.$$

with $w^*/|w|$ having magnitude 1, so that $w^*/|w| = e^{i\phi}$, for $\tan\phi = (\mathrm{Im}\ w^*)/(\mathrm{Re}\ w^*) = -(\mathrm{Im}\ w)/(\mathrm{Re}\ w) = \gamma\omega/(\omega^2 - \omega_0^2)$

and $|w| = [(\omega_0^2 - \omega^2)^2 + (\omega\gamma)^2]^{1/2}$

If a_0 is real, then

$$x = a_0 \cos(\omega t + \phi)\ [(\omega_0^2 - \omega^2)^2 + (\omega\gamma)^2]^{-1/2}.$$

If $\omega = \omega_0$, the amplitude is a maximum, $|A| = a_0/\omega_0\gamma$.

Forced Harmonic Oscillator

Without damping, the resonant amplitude would be infinite. What this means is the amplitude keeps growing forever, with nothing to slow it down.

You can check by substituting into the equation that if $\gamma=0$, we would have a growing solution

$z = -\frac{1}{2}(ia_0t/\omega_0)\, e^{i\omega\ t}$ or $x = \frac{1}{2}(a_0t/\omega_0)\sin\omega_0t$ which never reaches a steady state. Proof:

$\dot{x} = \frac{1}{2}a_0/\omega_0 \sin w_0t + \frac{1}{2}a_0t\cos\omega_0t$

$\ddot{x} = \frac{1}{2}a_0\cos\omega_0t + \frac{1}{2}a_0\cos\omega_0t - \frac{1}{2}a_0\omega_0t\sin\omega_0t.$

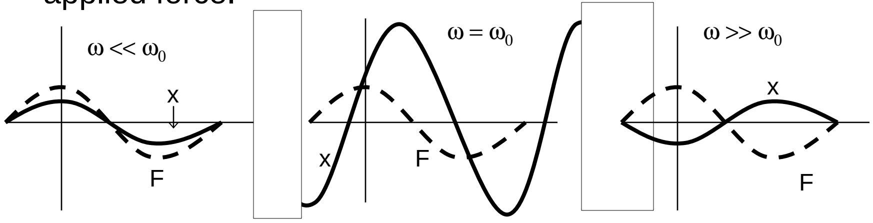
$\ddot{x} + \omega_0^2x = a_0\cos\omega_0t = F(t)/m.$

Forced Harmonic Oscillator

The phase factor of the response is given by (for a_0 real)

$$\tan\phi = \omega\gamma / (\omega_0^2 - \omega^2).$$

For very slow pumping, the response is approximately in phase, but begins to lag as the speed of pumping increases. At resonance, the force and response are 90° out of phase, with the force leading. That means the greatest force is always applied when the velocity is greatest, maximizing the power input P = Fv. For very fast ω, the response becomes 180° out of phase with the applied force.



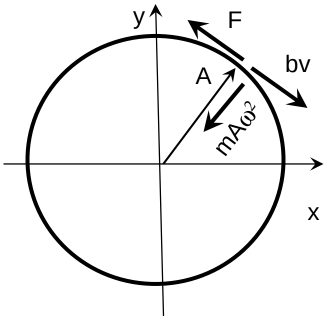
Forced Harmonic Oscillator

- The most effective pumping frequency is $\omega = \omega_0$. Then

$$F = m\, a_0 e^{i\omega t}$$
$$= mA\, i\gamma\omega e^{i\omega t} = bv.$$

The applied force just cancels the viscosity in this case in a steady state. Note that in the circular picture, the force is tangential to the circular motion in this case.

The “spring” is the centripetal force in this picture.



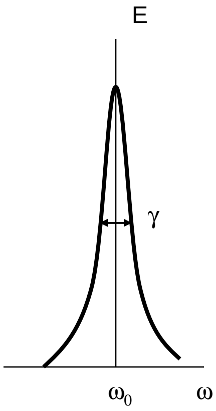
Narrow Resonance

When $\gamma << \omega_0$, (very underdamped), A_0 is very small away from the resonance, and ω_0, you can approximate $\omega_0^2 - \omega^2 \approx 2\omega_0(\omega_0 - \omega)$ and $\omega\gamma \approx \omega_0\gamma$.

$$A_0 = a_0 \left[(\omega_0^2 - \omega^2)^2 + (\omega\gamma)^2 \right]^{-1/2}$$
$$\approx a_0 \left[4\omega_0^2(\omega_0 - \omega)^2 + \omega_0^2\gamma^2 \right]^{-1/2}$$
$$= (a_0/2\omega_0)\left[(\omega_0 - \omega)^2 + (\gamma/2)^2\right]^{-1/2}$$

In this approximation, A_0 is symmetric in ω about ω_0, with the width of the response curve determined by γ.

Narrow Resonance Energy



The energy is $E = ½ kA_0^2$.

$$E \approx (ma_0^2/8)[(\omega_0 - \omega)^2 + (\gamma/2)^2]^{-1}$$

The maximum energy is

$$E_{max} = ma_0^2/2\gamma^2.$$

The energy drops to half when

$$\omega = \pm \gamma/2.$$

General Solution

Where did the free parameters go?

You can add on any solution of the unforced equations.

$$\ddot{x} + \gamma\dot{x} + \omega_0^2\, x = a_0\, e^{i\omega t}$$

will still hold if

$$z = A\, e^{i\omega t} + A_1\, e^{-\gamma t/2}\, e^{i\omega_1 t} + B_1\, e^{-\gamma t/2}\, e^{-i\omega_1 t}$$

for A_1, B_1 chosen according to the initial conditions. The A_1, B_1 terms are transient and die out with 1/e time $\gamma/2$. (Only A_1 is needed, again.) The term we found gives the steady-state behavior after the transients die out.