Physics 103

Department of Physics
Princeton University

Precept Notes

S. Yost
November 28, 2007

Non-Harmonic Oscillators
Damped and Forced Oscillators
Part 1

Announcements

- This week, we will finish Chapter 14 on oscillation. Also see the posted notes on damped and forced oscillation. Read them before Monday.
- The midterm grades are supposed to be ready, but they may not be up yet. Keep checking.
- If you didn't do as well as you were hoping, you can see me about ways to improve your strategies in this course.
- Problem set 10a is up and due tonight!
- There is some practice with complex numbers in this set. We will be using them Monday to work some differential equations. They are a good way to represent the projected circular motion picture of simple harmonic motion in more complicated contexts.

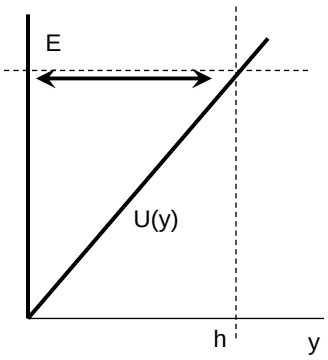
Anharmonic Oscillators

Not all oscillators are harmonic or even nearly harmonic.

- Example: Ball bouncing on a floor. Constant restoring force.

If the ball bounces perfectly from height h, then $h = \frac{1}{2} gt^2$ up or down, so the period is

$$T = 2t = 2(h/g)^{1/2}$$



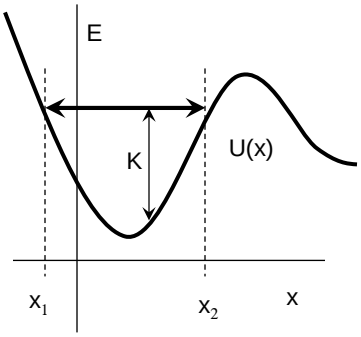
Period For General Oscillation

To find the period, we can calculate the time to get from x_1 to x_2 and double it.

$$T = 2 \int_0^{T/2} dt = 2 \int_{x_1}^{x_2} (dt/dx)dx$$
$$= 2 \int_{x_1}^{x_2} dx/v(x)$$

where v(x) can be found from energy conservation:

$$\frac{1}{2} mv^2 = E - U(x).$$



Period For Harmonic Oscillator

Example: Find T for the harmonic oscillator using the new method:

$$T = 2\int_{x_1}^{x_2} dx/v(x).$$

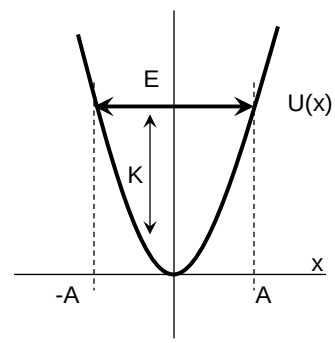
$$\tfrac{1}{2} mv^2 = \tfrac{1}{2} k(A^2 - x^2).$$

$$T = 2(m/k)^{1/2} \int_{-A}^{A} dx/(A^2 - x^2)^{1/2}$$

Trig substitution $x = A\sin\theta$.

$$dx = A\cos\theta d\theta,$$

$$(A^2 - x^2)^{1/2} = A\cos\theta.$$



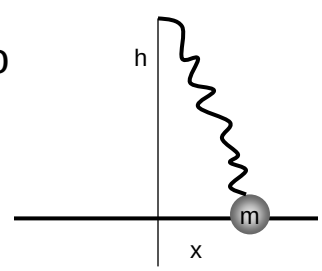
The integral just gives π! Then

$$T = 2\pi(m/k)^{1/2} = 2\pi/\omega$$

with $\omega = (k/m)^{1/2}$.

Anharmonic Oscillators

- Example: Mass attached to a spring of unstretched length h and constrained to slide on a track.



$$F_x = -k((x^2 + h^2)^{1/2} - h)x$$
$$(x^2 + h^2)^{1/2}$$

For $x << h$, $F_x = -\tfrac{1}{2} kx^3/h^2$.

$U(x) = -\int F_x dx = kx^4/(8h^2)$.

Anharmonic Oscillators

- How does T depend on A?

$\frac{1}{2} mv^2 = U(A) - U(x)$.

$dx/dt = (2/m)(U(A) - U(x))$.

$dt/dx = (m/2)^{1/2} (U(A) - U(x))^{-1/2}$

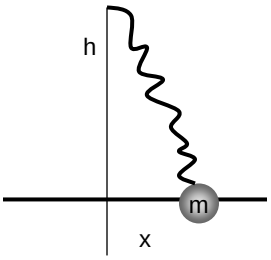
$T = 4(m/2)^{1/2}\int_0^A (U(A) - U(x))^{-1/2}\, dx$

$= 8h(m/k)^{1/2} \int_0^A (A^4 - x^4)^{-1/2}\, dx$

It would be good to get A out of the integral. Set $x = Au$, $dx = Adu$.

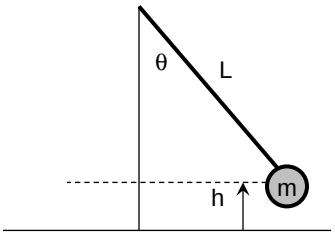
$T = (8h/A)(m/k)^{1/2} \boxed{\int_0^1 (1 - u^4)^{-1/2}\, du}$

Pure numerical constant = 1.311.
(not too important for interpreting the answer – just a number)



The period is inversely proportional to the amplitude.

Pendulum



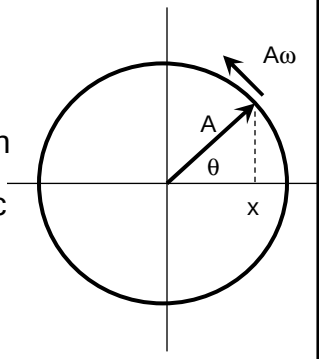
$U = mgh = mg(L - L\cos\theta)$

$K = \frac{1}{2} mL^2 (d\theta/dt)^2$

$(d\theta/dt)^2 = (2g/L)(\cos\theta - \cos\theta_0)$

Approximate form for small angles:

$(d\theta/dt)^2 \approx (g/L)(\theta_0^2 - \theta^2)$



This looks like the relation between position and velocity for a harmonic oscillator. To make it clearer, write it in terms of arc length $s = L\theta$:

$$v^2 = (g/L)(s_0^2 - s^2).$$

Compare $v_x = -A\omega \sin\theta = -\omega(A^2 - x^2)^{1/2}$ to see that $v^2 = \omega^2(A^2 - x^2)$ for SHM.

Then $\omega = (g/L)^{1/2}$ for the pendulum.

Pendulum

The exact integral for the period can be obtained from

$dt/d\theta = (2g/L)^{-1/2}(\cos\theta - \cos\theta_0)^{-1/2}$

$T = 4(L/2g)^{1/2} \int_0^{\theta_0} (\cos\theta - \cos\theta_0)^{-1/2}\, d\theta$

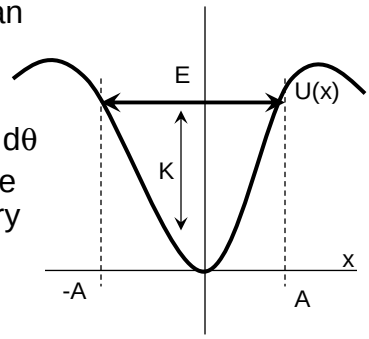
This is an elliptic integral. It can't be expressed in terms of elementary functions.

For $\theta_0 << 1$ rad (57°), the small amplitude approximation gives

$T \approx 4(L/g)^{1/2} \boxed{\int_0^{\theta_0} d\theta/(\theta_0^2 - \theta^2)^{1/2}}$

$= 2\pi(L/g)^{1/2}$.

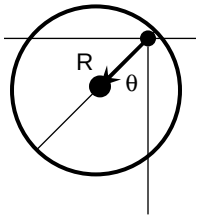
$\pi/2$



Hoop on a Nail

An example we didn't do in class but could have.

- What is the period of a hoop of radius R swinging (without slipping) on a nail? Assume a small amplitude of oscillation.



$U = mgR(1 - \cos\theta) \approx \frac{1}{2} mgR\theta^2$.

$K = \frac{1}{2} I_{nail}\omega^2$

$= mR^2 (d\theta/dt)^2$. $I_{nail} = 2mR^2$ (parallel axis theorem)

$(d\theta/dt)^2 = (g/2R)(\theta_0^2 - \theta^2)$

Recognize hramonic oscillator: $\omega^2 = g/2R$.

Complex Arithmetic

Complex arithmetic works like regular arithmetic (commutative, distributive, associative laws…) with $i^2 = -1$.

Multiplication:

$$(a+ib)(c+id) = ac - bd + i(bc + ad).$$

Complex conjugate: $z^* = x - iy$ if $z = x + iy$.

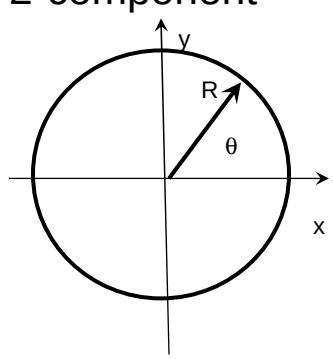
Magnitude: $|z| = (x2 + y2)^{1/2} = (zz^*)^{1/2}$.

When dividing, it is convenient to use the fact that $zz^* = |z|^2$, so that $w/z = wz^*/|z|$.

Geometry of Complex Numbers

Complex numbers are 2-dimensional: $z = x + iy$ is equivalent to a 2-component vector.

$e^{i\theta}$ is a unit vector at angle θ:

$$e^{i\theta} = \cos\theta + i \sin\theta.$$

$z = Re^{i\omega t}$ is a point rotating about the origin at radius R with angular velocity ω.