Designing Cyclic Universe Models

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The phenomenological constraints on the scalar field potential in cyclic models of the universe are presented. We show that cyclic models require a comparable degree of tuning to that needed for inflationary models. The constraints are reduced to a set of simple design rules including “fast-roll” parameters analogous to the “slow-roll” parameters in inflation.

The cyclic universe model \([?\) ] is a proposal addressing the homogeneity, flatness and monopole problems of the standard hot big bang picture, and also providing a nearly scale invariant spectrum of density fluctuations, without invoking a period of high energy inflation \([?\) ]. Each cycle consists of: (i) a hot big bang phase during which large-scale structures form, (ii) a phase of slow, accelerated expansion \([?\) ], as observed today, which dilutes inhomogeneities and flattens the universe, setting up two of the needed conditions for the next cycle, (iii) a phase of contraction (in Einstein frame) during which nearly scale invariant density perturbations are generated, and (iv) a big crunch/big bang transition at which matter and radiation are created and the next cycle is triggered.

The distinctive, non-inflationary mechanism for generating density perturbations in cyclic models results in a key observational difference: whereas inflation predicts a nearly scale invariant spectrum of gravitational waves, the cyclic model does not. The cyclic universe model is an extension of the ekpyrotic scenario \([?\) ? \)], in which the hot big bang is viewed as the result of a collision between two brane worlds, in the simplest case between two orbifold planes. The theory can be described by an effective 4d action in which the size of the orbifold is represented by a scalar field \(\phi\) and the force between the boundary planes is represented by an effective potential \(V(\phi)\). The cyclic model corresponds to regularly repeating collisions with a period of dark energy domination during each cycle. The condition that cycles repeat and that the resulting solution is an attractor requires that \(V(\phi)\) takes the general form shown in Fig. 1.

The purpose of this Letter is to summarize the constraints for designing successful cyclic models. We show that a wide range of scalar field effective potentials are phenomenologically viable. The constraints on the steepness of the potential turn out to be remarkably similar to those on the flatness of the potential in inflation. The constraints depend strongly on the amplitude of the growing mode density perturbation propagating across the bounce into the hot big bang phase. We employ here the recent treatment of the transition as a collision between branes in five dimensions \([?\) ], which resolves ambiguities present in earlier treatments\([?\) ? ? \]. Detailed derivations of the constraints are presented in a companion paper \([?\) ]; the gravity wave spectrum constraints quoted here are derived in \([?\) ].

At all times except around the big crunch/big bang transition, the dynamics of the cyclic model are well described by the 4d effective Einstein-frame Lagrangian

\[
\mathcal{L} = -\sqrt{-g} \left( \frac{\mathcal{R}}{2} - \frac{\left(\partial \phi\right)^2}{2} - V(\phi) - \beta^4(\phi)(\rho_M + \rho_R) \right),
\]

where \(g\) is the determinant of the metric \(g_{\mu\nu}\), \(\mathcal{R}\) the corresponding Ricci scalar, and we use units where \(8\pi G = 1\). The coupling \(\beta(\phi)\) depends on the details of the theory, but, when the branes approach each other, the warping of the extra dimension becomes irrelevant and one finds a universal behavior \(\beta \to \exp(-\phi/\sqrt{6})\) as \(\phi \to -\infty\). This limiting form ensures that the matter \((\rho_M)\) and radiation \((\rho_R)\) energy densities remain finite at the big crunch/bang \([?\) ].

The potential \(V(\phi)\) in the cyclic model, sketched in Fig. 2?, can be divided into three regions: (a) the range where \(V > 0\), (b) the range where \(V < 0\) is nearly exponential and steep, (c) the range where \(V\) deviates from this exponential steepness. Dotted lines indicate different allowed shapes for \(V\), illustrating the large freedom in the form of \(V\) in region (c). At the present epoch, the field lies at \(V = V_0\) (indicated by a dark circle) and its potential energy is the dark energy driving today’s observed acceleration of the universe (Region (a)). After a period
of acceleration dilutes away inhomogeneities and flattens the universe, the field rolls down towards negative $V$, and the acceleration terminates. Eventually, as the potential becomes negative, Einstein-frame expansion reverses to contraction (Region (b)). The large negative curvature of the potential in this region leads to an instability amplifying quantum mechanical vacuum fluctuations in $\phi$ into a classical, nearly scale invariant spectrum of perturbations. As Einstein-frame contraction speeds up, $\phi$ rolls off to minus infinity in finite time (Region (c)) and the universe undergoes a big crunch/big bang transition. The brane collision generates matter and radiation, with the fluctuations produced in Region (b) being imprinted as curvature perturbations in the ensuing hot big bang phase, a process described in detail from the 5d point of view in Ref. [? ]. After the collision, as the branes separate $\phi$ rebounds from minus infinity and due to Hubble damping, settles somewhere in the range $V > 0$. After the standard radiation- and matter-dominated epochs, the positive potential energy once again triggers cosmic acceleration. The cycle then repeats itself.

As indicated in Fig. ??, there is tremendous flexibility in the shape of $V(\phi)$ in Regions (a) and (c). Most of the quantitative constraints apply to Region (b), where density perturbations are generated. Even here, once four primary constraints (labeled (i)-(iv) below) are satisfied, the remaining requirements which we discuss are generally satisfied as well.

**Region (b): Spectral index of density perturbations.**

Scale invariant fluctuations can be generated during a contracting phase if the energy density in $\phi$ dominates, and if its equation of state parameter is much greater than unity and nearly constant, as shown by Gratton et al. [? ]. A nearly scale-invariant spectrum is obtained if $V < 0$ and $\epsilon \equiv (V/V_\phi)^2 \ll 1$ and $|\eta| \equiv |1 - (V/V_\phi)/V_\phi^2| \ll 1$, where $V_\phi = dV/d\phi$. Qualitatively, these require, respectively, that $V$ be negative, very steep and nearly exponential in form. These are almost the exact opposite of the slow-roll conditions of inflation which require $V$ to be positive and flat. In particular, $\epsilon$ and $\eta$ should be thought of as “fast-roll” parameters, in analogy with the inflationary slow-roll parameters: $\epsilon \equiv V_\phi^2/2V^2$ and $\eta \equiv V_{\phi\phi}/V$.

As the universe emerges from the era of slow accelerated expansion, $H^2 \ll -V$, i.e., the kinetic and potential energy of $\phi$ nearly cancel out. More precisely, the cosmological evolution in phase (iii) is described by [? ]

\[ a \sim (-\tau)^{2\epsilon}, \quad H \approx -\sqrt{-2\epsilon V}, \quad \phi' \approx \sqrt{4\epsilon/\tau}, \quad (2) \]

where $\tau < 0$ is conformal time and prime denotes $\tau$ derivative. Since $\epsilon \ll 1$, the scale factor varies very slowly during this phase.

As in inflationary cosmology, the spectral index can be neatly expressed in terms of $\epsilon$ and $\eta$. Using Eq. (??), the equation for the Fourier mode $k$ of the gauge-invariant variable $u$, related to the Newtonian potential $\Phi$ by $u = a\Phi/\phi'$, reduces to [? ]

\[ u_k' + \left[ k^2 - \frac{2(\epsilon + \eta)}{\tau^2} \right] u_k = 0, \quad (3) \]

from which one can read off the spectral index

\[ n_s - 1 = -4(\epsilon + \eta). \quad (4) \]

The analogous equation for inflation leads to the familiar result $n_s - 1 = -6\epsilon + 2\eta$.

In deriving Eq. (??), it is assumed that $\epsilon$ and $\eta$ are nearly constants for observable modes. For a spectral range of $N \sim 60$ e-folds, say, constancy of $\epsilon$ requires $|\eta| < 1/4N$ [? ]. Combining this with the observational constraint $|n_s - 1| \leq 0.1$, we find

\[ (i) \quad \epsilon \lesssim 1/40, \quad |\eta| \lesssim 1/240. \quad (5) \]

The same analysis for inflation results in nearly identical constraints, $\epsilon \lesssim 1/60$ and $\eta \lesssim 1/120$.

**Amplitude of density perturbations.**

Assuming that fluctuations in $\phi$ start in their Minkowski vacuum, when $k^2\tau^2 \gg 2(\epsilon + \eta)$, the solution to Eq. (??) in the long-wavelength limit is

\[ k^{3/2} \Phi_k \approx 2^{-3/2} \left( \frac{\phi'}{a} \right) \left( -\frac{k\tau}{2} \right)^{-2(\epsilon + \eta)}. \quad (6) \]

This expression holds provided $\epsilon$ and $\eta$ are small, which we assume is true until $\phi$ approaches the point $\phi_{end}$ where $V = V_{end}$ in Fig. ??, The last factor is of order unity and only weakly $k$-dependent for small $\epsilon$ and $\eta$, hence we ignore it for the rest of this section. Rewriting the velocity of the scalar field in terms of the potential, we find $k^{3/2} \Phi_k \approx V_{end}^{1/2} / 2$ at $\phi \approx \phi_{end}$. As $\phi$ moves past $\phi_{end}$ its evolution becomes kinetic-dominated and the potential becomes irrelevant. For simplicity we analyze the case where the potential climbs rapidly to zero for $\phi < \phi_{end}$. The constraints for other cases are only slightly altered.

In the kinetic-dominated phase, the fractional energy density perturbation on comoving slices, $\epsilon_m \equiv 2k^2\Phi_k/3H^2a^2$, is for small $k$ a constant $\epsilon_0$ all the way to the singularity $\tau = 0$. We adopt the same normalization for the scale factor $a$ as that of Ref. [? ], according to which $a^2 \approx 2v_{col}^4/\tau L$ (see Section IIIIC of that paper), where $v_{col}$ is the relative speed (which we shall assume nonrelativistic) of the two colliding branes and $L$ is the curvature scale associated with the warping of the extra dimension (in heterotic M-theory models[? ?, for example, typical values lie in the range $L = 10^6-6$]). In the kinetic dominated phase we have $H^2a^2 = \frac{1}{2} \dot{\phi}^2 = \frac{1}{2} \tau^{-2}$. We solve for $\tau_{end}$, the conformal time at the start of the kinetic-dominated phase, by equating the kinetic energy $\dot{\phi}^2/(2a^2)$ to the value determined from energy conservation as $\phi$ passes the potential step, $6\epsilon V_{end}$ obtaining

\[ k^3 \epsilon_0 \approx \frac{k^2 L^4}{3V_{end}^2 \tau_{end}^2 v_{col}^2}. \quad (7) \]
From Ref. [? ], the long wavelength curvature perturbation on comoving slices following the boundary brane collision is then given, for non-relativistic $v_{\text{coll}}$, by

$$k^{3/2} \zeta \sim \frac{3v_{\text{coll}}^0}{32L^2 k^{1/2}} \sim \frac{1}{32} \frac{v_{\text{coll}}^{10/3}}{\epsilon^{2/3} L^{4/3} V_{\text{end}}^{1/6}}.$$  \hfill (8)

A second radiation-dependent contribution is also discussed in Ref. [? ], but this is always sub-dominant if the radiation density on the branes is small compared to the brane kinetic energies at collision (see equation (??) below).

Measurements of the cosmic microwave background (CMB) constrain $k^{3/2} \zeta$ to be $10^{-5}$, which implies

$$V_{\text{end}}^{1/4} \approx 10^5 v_{\text{coll}}^5 \epsilon^{-1} L^{-2}. \hfill (9)$$

**Bounds on the radiation density.** Although it is not necessarily required, we will consider the case where the radiation energy density produced at the brane collision is smaller than the magnitude of the brane kinetic energies ($\sim v_{\text{coll}}^2 L^{-2}$). From the formulae of Section III C of Ref. [? ], we infer the Hubble constant $H_r$ at radiation-kinetic equal density, $H_r \sim T_r^2$, obtaining

$$T_r \lesssim \sqrt{v_{\text{coll}}/L}. \hfill (10)$$

This constraint ensures that the radiation-dependent contribution to $\zeta$ mentioned following equation (??) above is small compared to the radiation-independent contribution considered here.

The temperature at the beginning of the radiation-dominated epoch should exceed $\sim 1$ MeV to recover the successful predictions of the hot big bang nucleosynthesis. This implies

$$(ii) \quad T_r \approx H_r^{1/2} > 10^{-21}. \hfill (11)$$

**Cycling constraint.** Once the radiation-dominated epoch begins, the scalar field kinetic energy is rapidly damped and the field comes to halt. In order to cycle, the radiation-dominated epoch should not begin until $\phi$ has time to go from $-\infty$, past $V = -V_{\text{end}}$, right across the potential well and back up the potential to the plateau where $V = V_0$. During this period, the scalar field kinetic energy density is dominant and $-\phi \sim \sqrt{2/3} \ln(t/t_{\text{end}})$, where $t_{\text{end}}$ is the time it takes $\phi$ to reach $V = -V_{\text{end}}$ after the brane collision. Assuming an exponentially steep potential as in Ref. [? ] with $V \approx V_0 (1 - \exp(-c\phi))$ (corresponding to $\epsilon = e^{-2}$ in the potential well), the time required to climb from $V \approx -V_{\text{end}}$ to $V_0$ and begin the radiation-dominated epoch is $t_r > t_{\text{end}}(V_{\text{end}}/V_0)^{\sqrt{3/2}}$. Equality between scalar field kinetic energy and radiation occurs at $t_r/t_{\text{end}} \approx V_{\text{end}}^{1/2}/H_r$. Consequently, we have the constraint

$$(iii) \quad T_r \lesssim V_{\text{end}}^{1/4} (V_0/V_{\text{end}})^{\sqrt{3/8}}. \hfill (12)$$

**Gravitational wave bound.** The calculation of the spectrum of gravity waves is very similar to that for scalar perturbations presented above. A systematic analysis is given in Ref. [? ]. The most stringent requirement is that the gravitational wave contribution to the total energy density of the universe be less than 10% of the radiation density at nucleosynthesis; this results in the constraint

$$(iv) \quad T_r \gtrsim 10^{14}/V_{\text{end}}. \hfill (13)$$

**Spectral range of the fluctuations.** The perturbations produced in the ekpyrotic phase as $\phi$ rolls from $V \approx 0$ to $V_{\text{end}}$ is $k_{\text{max}}/k_{\text{min}} \approx \sqrt{V_{\text{end}}/V_0}$. To span 60 e-folds, for example, we require

$$V_0^{1/4} < e^{-30} V_{\text{end}}^{1/4} \approx 10^{-13} V_{\text{end}}^{1/4}. \hfill (14)$$

This does not explain why the dark energy is so small, but it does imply that an exponential hierarchy of scales is necessary in order for the perturbations to be nearly scale invariant over a broad range of scales.

As illustrated in Figure 2, a very broad range of parameters satisfies all of the conditions in Region (b).

**Region (a):** In the range $V > 0$, there is a lot of freedom for different designs. First, the value of $V$ today ($V_0$), must equal $10^{124}$ in order to account for the observed dark energy. This fine-tuning should not be viewed as a disadvantage relative to inflation since the dark energy must be explained in that scenario as well. Second, the potential must join onto the steep negative part by either a plateau, as in Fig. 1, or an energy barrier as will be discussed elsewhere. In the former case, the potential must be sufficiently flat for the current acceleration to be sustained for several tens of e-foldings, enough to smooth out the universe and reduce the perturbations to a level insignificant compared to the new fluctuations produced in the subsequent contracting phase. If the number of e-foldings is greater than 60 (about 3 trillion years), the current particle density ($10^{80}$ particles per Hubble volume) is reduced to less than one per Hubble volume. This is more than enough to ensure the universe is made homogeneous, isotropic and flat. The condition is relatively easy to satisfy, as illustrated in most quintessence models [? ]. The form of $V$ for $\phi$ greater than the current value is not important for the phenomenology of the cyclic model, although it may be in determining whether the cyclic solution is a global attractor. Third, $\phi$ must not violate precision tests of general relativity. In the case of the plateau where the field is nearly massless, this can be avoided by having the couplings to matter obey $d\ln \beta/d\phi < 10^{-3}$ for $\phi$ near the current value [? ]. In the case of an energy barrier, the field is currently trapped at a local minimum and its mass can easily be made sufficiently large to avoid equivalence principle violations on observable scales.

**Region (c):** The only constraint in this region is that the kinetic energy in $\phi$ must dominate the potential energy near the big crunch/big bang transition ($\phi \to -\infty$), a necessary condition for a bounce according to the prescription in Ref. [? ]. This will be the case if $d^3 V \rightarrow 0$
as \( \phi \to -\infty \). For example, potentials in which \( V(\phi) \) approaches a constant or monotonically decreases (more slowly than \( 1/a^6 \)) as \( \phi \to -\infty \) satisfy this condition. An example is \( V(\phi) \propto -e^{-c\phi} \) with \( c < \sqrt{6} \). The M theory setup motivates one to consider \( \phi_H \) potentials where \( V \to 0 \) as \( \phi \to -\infty \) since M theory reduces to a weakly coupled string theory in this limit. However there does not seem to be any such requirement imposed from the phenomenology.

The tightest constraints on the model emerge from Region (b), where density perturbations are created. Similarly, in inflationary cosmology, only the region of the potential involved in fluctuation generation is tightly constrained.

For practical purposes, only the four constraints labeled \((i)-(iv)\) impose any significant restrictions on cyclic model building. Constraint \((i)\) (Eq. ??) imposes a constraint on the steepness of the potential expressed through fast-roll parameters \( \epsilon \) and \( \eta \) that are analogous to inflationary slow-roll parameters. Constraints \((ii)-(iv)\) (Eqs. ??, ??, and ??) limit the remaining physical parameters.

As illustrated in Fig. 2, a very wide range of physically plausible parameters is allowed, broadly similar to that for inflationary models. Hence in the absence of deeper theoretical constraints on the form of potentials, and parameters such as the brane speeds at collision, it is premature to claim a significant advantage for either model in terms of tuning.

We would however argue that since the cyclic model requires only one epoch of accelerated expansion, driven by today’s observed vacuum energy, it is inherently more economical than inflation. Standard inflationary models require two independent accelerated phases, with corresponding dark energy densities differing by around one hundred orders of magnitude.

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