Cosmological Considerations in the General Theory of Relativity

This translation by W. Perrett and G. B. Jeffery is reprinted from H. A. Lorentz et al., The Principle of Relativity (Dover, 1952), pp. 175–188.

It is well known that Poisson’s equation
\[ \nabla^2 \phi = 4\pi G \rho, \quad \text{(1)} \]
in combination with the equations of motion of a material point is not as yet a perfect substitute for Newton’s theory of action at a distance. There is still to be taken into account the condition that at spatial infinity the potential \( \phi \) tends toward a fixed limiting value. There is an analogous state of things in the theory of gravitation in general relativity. Here, too, we must supplement the differential equations by limiting conditions at spatial infinity, if we really have to regard the universe as being of infinite spatial extent.

In my treatment of the planetary problem I chose these limiting conditions in the form of the following assumption: it is possible to select a system of reference so that at spatial infinity all the gravitational potentials \( \phi \) become constant.

But it is by no means evident a priori that we may lay down the same limiting conditions when we wish to take larger portions of the physical universe into consideration. In the following pages the reflections will be given which, up to the present, I have made on this fundamentally important question.

§ 1. The Newtonian Theory

It is well known that Newton’s limiting condition of the constant limit for \( \phi \) at spatial infinity leads to the view that the density of matter becomes zero at infinity. For we imagine that there may be a place in universal space round about which the gravitational field of matter, viewed on a large scale, possesses spherical symmetry. It then follows from Poisson’s equation that, in order that \( \phi \) may tend to a
It seems hardly possible to surmount these difficulties on the basis of the Newtonian theory. We may ask ourselves the question whether they can be removed by a modification of the Newtonian theory. First of all we will indicate a method which does not in itself claim to be sound seriously: it merely serves as a lot for what is to follow. In place of Poisson's equation we write

$$\nabla \phi = -4\pi G \rho$$

where $\rho$ denotes a universal constant. If $\rho$, be the uniform density of a distribution of mass, then

$$\phi = -\frac{4\pi G \rho}{r}$$

is a solution of equation (1). This solution would correspond to the case in which the matter of the fixed stars was distributed uniformly through space, if the density $\rho$, is equal to the actual mass density of the matter in the universe. The solution then corresponds to an infinite extension of the central space, filled uniformly with matter. If, without making any change in the mass density, we imagine matter to be non-uniformly distributed locally, there will be, over and above the $\phi$ with the constant value of equation (5), an additional $\phi$, which in the neighborhood of dense masses will so much the more resemble the Newtonian field as $\alpha$ is smaller in comparison with $4\pi G \rho$. A universe so constituted would have, with respect to its gravitational field, no center. A decrease of density in spatial infinity would not have to be assumed, but both the mean potential and mass density would remain constant to infinity. The conflict with observational mechanics which we found in the case of the Newtonian theory is not removed. With a density but extremely small density, matter is in equilibrium, without any internal material forces (pressures) being required to maintain equilibrium.

§ 3. The Boundary Conditions According to the General Theory of Relativity

In the present paragraph, I shall conduct the reader over the road that I have myself travelled, rather a rough and winding road, because otherwise I cannot hope that he will
for the components of momentum, and for the energy (in the static case) $m/\sqrt{B}$.

From the expressions for the momentum, it follows that $\sqrt{B}$ plays the part of the rest mass. As $m$ is a constant peculiar to the point of mass, independently of its position, this expression, if we retain the condition $\sqrt{B} = 1$ at spatial infinity, can vanish only when $A$ diminishes to zero, while $B$ increases to infinity. It seems, therefore, that such a degeneration of the coefficients $\mu_i$ is required by the postulates of relativity of all inertias. This requirement implies that the potential energy $m/\sqrt{B}$ becomes infinitely great at infinity. Thus a point of mass can never leave the system; and a more detailed investigation shows that the same thing applies to light-rays. A system of the universe with such behavior of the gravitational potentials at infinity would not therefore rev the risk of wasting away which was mooted just now in connection with the Newtonian theory.

I wish to point out that the simplifying assumptions as to the gravitational potentials on which this reasoning is based, have been introduced merely for the sake of facility. It is possible to find general formulations for the behavior of the $\phi_i$ at infinity which express the essentials of the question without further restrictive assumptions.

At this stage, with the kind assistance of the mathematician J. Stummon, I investigated centrally symmetry, static gravitational fields, degenerating at infinity in the way mentioned. The gravitational potentials $\phi_i$ were applied, and from them the energy-tensor $T^\mu_\nu$ of matter was calculated on the basis of the field equations of gravitation. But here it proved that for the system of the fixed stars no boundary conditions of the kind one can come into question at all, as was also rightly emphasized by the astronomer de Sitter recently.

For the contravariant energy-tensor $T^\nu_\mu$ of ponderable matter is given by

$$T^\nu_\mu = p \frac{d^\nu x}{dx^\mu},$$

where $p$ is the density of matter in natural measure. With
In the first place those boundary conditions presuppose a definite choice of the system of reference, which is contrary to the spirit of the relativistic principles. Secondly, if we adopt this view, we fail to comply with the requirement of the relativistic character. For the inertia of a material point of mass $m$ (in natural measure) depends upon the $g_{00}$, but these differ but little from their postulated values, as given above, for spatial infinity. Thus inertia would indeed be influenced, but would not be conditioned by matter (present in finite space). If only one single point of mass were present, according to this view, it would possess inertia, and in fact an invariant almost as great as when it is surrounded by the other masses of the actual universe. Finally, these statistical objections must be raised against this view which were mentioned in respect of the Newtonian theory.

From what has now been said it will be seen that I have not succeeded in formulating boundary conditions for spatial infinity. Nevertheless, there is still a possible way out, without resigning as suggested under (6). For if it were possible to regard the universe as a continuum which is finite (closed) with respect to its spatial dimensions, we should have no need at all of any such boundary conditions. We shall proceed to show that both the general postulate of relativity and the fact of the small stellar velocities are compatible with the hypothesis of a spatially finite universe; though certainly, in order to carry through this idea, we need a generalizing modification of the field equations of gravitation.

§ 3. The Spatially Finite Universe with a Uniform Distribution of Matter

According to the general theory of relativity the material character (curve of the four-dimensional space-time continuum is defined at every point by the matter at that point and the state of that matter. Therefore, on account of the lack of uniformity in the distribution of matter, the material structure of this continuum must necessarily be extremely complicated. But if we are concerned with the structure only on a large scale, we may represent matter to ourselves as being uniformly distributed over enormous space, so that its density of distribution is a variable function which varies...
extremely short. Thus our procedure will somewhat resemble that of the geodesics, who, by means of an ellipsoid, approximate to the shape of the earth's surface, which on a small scale is extremely complicated.

The most important fact we draw from experience as to the distribution of matter is that the relative velocities of the stars are very small as compared with the velocity of light. So I think that for the present we may base our reasoning upon the following approximative assumption. There is a system of reference relative to which matter may be looked upon as being permanently at rest. With respect to this system, therefore, the contradistinction concerning $T^m_{mn}$ of matter is, by reason of (5), of the simple form

$$\begin{align*}
\rho & = 0 \\
\rho & = 0 \\
\rho & = 0 \\
\rho & = 0
\end{align*}$$

The scale $\rho$ of the (mean) density of distribution may be a priori a function of the space coordinates. But if we assume the universe to be spatially finite, we are compelled to the hypothesis that $\rho$ be independent of location. On this hypothesis we base the following considerations.

As concerns the gravitational field, it follows from the equations of motion of the material point

$$d^2x = \left\{\begin{align*}
e0, \\
0, \\
0, \\
ne0
\end{align*}\right.$$
which equation, in combination with (7) and (8), perfectly defines the behavior of measuring-rod, clocks, and light-rays.

§ 4. An Additional Term for the Field Equations of Gravitation

My proposed field equations of gravitation for any chosen system of co-ordinates takes as follows:

$$\nabla^2 u = - \left( \nabla^2 u + (\mu, T) \right)$$

$$\text{(13)}$$

$$\nabla^2 u = - \frac{1}{2} \left[ \frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{\partial r} \right] + \left[ \frac{\partial u}{\partial x} \right]^2 + \left[ \frac{\partial u}{\partial y} \right]^2 + \left[ \frac{\partial u}{\partial z} \right]^2$$

The system of equations (13) is by no means satisfied when we insert for the $\mu$, the values given in (7), (8), and (12), and for the (contravariant) energy-tensor of matter the values indicated in (6). It will be shown in the next paragraph how this calculation may conveniently be made. So that, if it were certain that the field equations (13) which I have hitherto employed were the only ones compatible with the principles of general relativity, we should probably have to conclude that the theory of relativity does not admit the hypothesis of a spatially finite universe.

However, the system of equations (14) allows a readily suggested extension which is compatible with the relativity principle, and is perfectly analogous to the extension of Poisson's equation given by equation (5). For on the left-hand side of field equation (13) we may add the fundamental tensor $\mu_{\alpha\beta}$ multiplied by a universal constant $\lambda$ at present unknown, without destroying the general covariance. In place of field equation (13) we may write

$$\nabla^2 u = - \left( \nabla^2 u + (\mu, T) \right) + \lambda$$

$$\text{(13a)}$$

This field equation, with $\lambda$ sufficiently small, is in any case also compatible with the facts of experience derived from the solar system. It also satisfies laws of conservation of momentum and energy, because we arrive at (13a) in place of (13) by introducing into Hamilton's principle, instead of the scalar of Riemann's tensor, this scalar increased by a universal constant; and Hamilton's principle, of course, guarantees the validity of laws of conservation. It will be shown in § 5 that field equation (13a) is compatible with our conjectures on field and matter.

§ 5. Calculation and Torsion

Since all points of our continuum are on an equal footing, it is sufficient to carry through the calculation for one point, e.g. for one of the two points with the coordinates $x_1 = x_2 = x_3 = 0$.

Then for the $\mu$, in (13a) we have to insert the values

$$\begin{align*}
\lambda & = 0 \\
\rho & = 0 \\
\theta & = 0 \\
\phi & = 0
\end{align*}$$

whenever they appear differentiated only once or not at all.

We then obtain in the first place

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

$$\text{(14)}$$

From this we readily discover, taking (7), (8), and (13) into account, that all equations (13a) are satisfied if the two relations

$$\begin{align*}
\lambda & = \frac{1}{3} \\
\lambda & = \frac{1}{3}
\end{align*}$$

$$\text{(14a)}$$

are satisfied.

Thus the newly introduced universal constant $\lambda$ defines both the mean density of distribution $\rho$ which can remain in equilibrium and also the radius $R$ and the volume $4\pi R^3$ of spherical space. The total mass $M$ of the universe, according to our view, is finite, and in fact

$$M = \rho \cdot 4\pi R^3 = \frac{4\pi R^3}{\rho} = \frac{\lambda}{\sqrt{\phi}}$$

$$\text{(15)}$$

Thus the historical view of the actual universe, if it is in consequence of our reasoning is the following: The
curvature of space is variable in time and place, according to the distribution of matter, but we may roughly approximate to it by means of a spherical space. At any rate, this view is logically consistent, and from the standpoint of the general theory of relativity has nearest at hand; whether, from the standpoint of present astronomical knowledge, it is tenable, will not here be discussed. In order to arrive at this consistent view, we admittedly had to introduce an extension of the field equations of gravitation which is not justified by our actual knowledge of gravitation. It is to be emphasized, however, that a positive curvature of space is given by our results, even if the supplementary term is not introduced. That term is necessary only for the purpose of making possible a quasi-static distribution of matter, as required by the fact of the small velocities of the stars.