Overall, I think the class did reasonably well on the midterm. If you have any questions about the grading or about the exam, you should feel free to contact your preceptor, or Prof. Steinhardt, or me (email poyang@princeton.edu). Very little credit was deducted for errors which were purely arithmetic in nature, and full (or very nearly full) credit was given for answers which were correct although not in simplest form. The following answers are generally more detailed than what we expected to see on the exam. – Peter

Section A: Short Answer (65 pts.)

1. (20 pts.) At noon today in Princeton, where was the Sun?

   This one I’m not going to answer. Go out and look! The whole point is to get you to become a better observer of your surroundings and more appreciative and curious about the surprises you may discover.

2. (15 pts.) Explain why there is a limit to how far you can see in the Universe, even with the largest telescopes.

   The ultimate limit on how far you can see (sometimes called a “horizon” in physicists’ jargon) exists because the universe has a finite age and because light has a finite velocity. There is some dispute about the precise age of the universe, but general consensus places the age between 12 and 20 billion years. Then because light has a finite velocity we can only see as far as the objects whose light has been traveling for long enough to reach us.

3. (15 pts.) Explain what is dark matter and how we know it exists.

   In its most general sense, dark matter is the matter in the universe which is not glowing (unlike stars, quasars, etc.) Thus we can’t see it in our telescopes. We know that it should exist by various indirect means. Although it is not glowing, dark matter still must exert a gravitational force if it has mass. In lecture, you heard about observations of the rates of rotation of galaxies. These observations found that, as one would expect from Kepler’s laws, the outer stars in galaxies have a velocity which depends on the distance of the stars from the galactic center as \( v \propto \sqrt{\frac{GM}{r}} \) and inferred the value of \( M \), the mass of the galactic center, from these observations. The mass you find this way is much more than the mass of the stars you can see in the galaxy! Even if you’re not yet convinced, there is much more evidence from other types of observations, such as gravitational bending of light around galaxies, and the motion of galaxies in clusters (due to Zwicky.) Exactly what is the dark matter, and what are its properties? That no one knows, although there are many suggestions (some rather exotic) from particle physics as to what it may be. This is a very exciting area of current research!

   Some students answered that dark matter is something which light cannot escape from, or which blocks light traveling from distant regions of the universe. This idea confuses dark matter and black holes, which are completely different things, but unfortunately enjoy very similar nomenclature. Dark matter does not interact (or interacts very weakly) with light; for practical purposes it neither absorbs nor scatters light. This is what makes dark matter so hard to observe. Black holes, on the other hand, can be detected by studying the luminous matter around them, and there is growing evidence that black holes do in fact exist at the centers of galaxies.

4. (15 pts.) If the Universe began with an explosion, would all galaxies move outward at the same speed (no matter what the mass)? Explain based on physical principles.
This question is related to a demonstration you saw in lecture. In that demo, you saw a simple two-body explosion, in which the two bodies had the same momentum but different masses, and hence different velocities (recall $p = mv$). If the universe began in such a way, it is highly highly implausible that they would be moving outward from the point of explosion with essentially the same speeds in all directions. The point of this question is to highlight that the “Big Bang,” as we currently understand it, was NOT an explosion, but rather a rapid stretching of space which happened *everywhere* in space, with the result that everything moved away from everything else with the same velocity (as an analogy, imagine marking points on the surface of a balloon and inflating the balloon. Every point moves away from every other point in a symmetric way.) You will hear more about the Big Bang in coming lectures.

Section B (235 pts.)

For this section, imagine that you travel on a mission to a distant solar system. The star is called Solus and the third planet from Solus is called Terra. Although the same laws of physics apply throughout the universe, the mass and size of Terra is different from Earth’s. As announced at the beginning of the exam, you are given that the radius of Terra is 3 times the radius of Earth.

1. (15 pts.) After you land on Terra, you drop a ball from the top of your rocket. It takes 3 seconds to reach the ground. On Earth, dropped from the same height, the ball takes 2 seconds to reach the ground. What is the acceleration of gravity near the surface of Terra? Show work.

Let $h$ be the distance from the top of your rocket to the ground, which is independent of whether you are on Earth or Terra. By “dropping” the balls, it is meant that their initial velocity is zero. From the standard kinematic formulas,

$$h = \frac{1}{2} g_E t_E^2 = \frac{1}{2} g_T t_T^2$$

where the subscripts refer to Earth and Terra in an obvious way. We are told that $t_E = 2$ s and $t_T = 3$ s so that

$$g_T = g_E \times \left( \frac{t_E}{t_T} \right)^2 = \frac{4}{9} g_E.$$  

The numerical value of $g_T$ is about $4.4 m/s^2$.

2. (10 pts.) You step on your bathroom scale to check your weight. You weigh 540 N on Earth, but the scale now reads 240 N. Is this consistent with what you learned from your experiment in Problem 1? Or did your scale break on the trip? Show your work and explain.

Your weight on Terra is $mg_T = \frac{4}{9} (mg_E)$. But $(mg_E)$ is just your weight on Earth, so your weight on Terra is $mg_T = \frac{4}{9} \times 540 N = 240 N$. This is consistent with the reading on the scale, which appears to have survived the journey just fine.

3. (20 pts.) You decided to do some further experiments:

(a) At the same instant that you drop the ball, you shoot a bullet horizontally at a speed of 200 m/s. Which hits the ground first, the bullet or the ball. (Neglect air resistance.) Explain your reasoning.

The bullet and the ball hit the ground at the same time. The vertical trajectory depends only on the acceleration ($g_T$) and the starting conditions (initial position and velocity.) These are all independent of mass. In fact it is
generally true that if you can ignore the motion of whatever supplies the gravitational field (in this case, you treat Terra as stationary) then Newton's equation looks like

\[ m\ddot{a} = -\frac{GMm}{r^2} \hat{e} \cdot r \hat{e}, \]

where \( \hat{e} \) is an appropriately directed vector of unit magnitude (it only indicates the direction of the force), and \( \hat{R} \), the position vector of the center of the planet, is constant. Note that little \( m \), the mass of the object which is moving, cancels from both sides of this equation.

(b) How far does the bullet travel before it hits the ground? Or do you need more information to answer the question? Show work/explain.

From Problem #1, you know that the bullet will take 3 seconds to hit the ground. In that time, it will travel \( d = v_y t = 200 \times 3 = 600 \) meters because there is no horizontal acceleration.

4. (15 pts.) From the ground, you shoot the bullet straight up at 200 m/s. How far up does it go? Show your work. [Use the answer you obtained for \( g \) from Problem 1, or, if you did not get an answer, pretend your answer was \( g = 20 \text{ m/s}^2 \).]

The easiest way to do this problem is by using conservation of energy. At the beginning of the motion, the bullet has kinetic energy equal to \( \frac{1}{2}mv^2 \). The bullet reaches its maximum height when all this kinetic energy is converted into potential energy, so if we call \( y \) the maximum height of the bullet,

\[ mgy = \frac{1}{2}mv^2 \Rightarrow y = \frac{v^2}{2g} = \frac{(200\text{ m/s})^2}{2 \times 4 \times 9.8\text{ m/s}^2} \approx 4500\text{ m}. \]

Of course, it's also possible to solve by using the kinematic equations for constant acceleration. The physics is the same any way you do it, but in this problem conservation of energy is simpler and easier to check for errors.

5. (15 pts.) You shoot two bullets along the two paths, A and B, in the figure. Which will land first? Or is it that you must have more information to answer the question? Credit is only given if the explanation based on physical principles is correct.

From the diagram, notice that the bullet fired along path A reaches a greater maximum height than the bullet on path B. Thus you know from conservation of energy that \( v_{yA} > v_{yB} \). At this point a qualitative argument that greater \( v_y \) means that the bullet takes longer to hit the ground was acceptable, but I will give a more quantitative argument. The condition for the bullet to hit the ground is that

\[ 0 = h + v_{y}t - \frac{1}{2}gt^2 \Rightarrow t = \frac{v_y}{gt} + \sqrt{\left(\frac{v_y}{gt}\right)^2 + \frac{2h}{gt}}. \]

Clearly \( t \), the time it takes for the bullet to hit the ground, increases with \( v_{y} \), implying that the bullet fired along path B will hit the ground first.

6. (10 pts.) Confident that you have not gained too much weight by coming to Terra, you climb up a rope to a cliff 20 m high to get a better view. How much work have you done? Show work.
Here you simply need to remember than work = force \times distance; simple multiplication is valid in this case because you are climbing straight up, and the force (gravity) is completely along the direction of motion (more generally, there might be forces perpendicular to the trajectory. These change the direction of motion but do no work.) Then you can substitute the given distance and the weight from Problem #2.

\[ W = F \times d = 240 \text{N} \times 20 \text{m} = 4800 \text{J}. \]  

(6)

7. (15 pts.) Based on what you have learned so far, is Terra more or less massive than the Earth? Explain how you came to this conclusion. By what factor? Show work.

Recall that for \( F_{\text{grav}} = mg \) on Terra, we must have \( g_T = \frac{GM_T}{R_T^2} \), and similarly for Earth. We also know that \( R_T = 3R_E \), and from Problem #1, \( g_T = \frac{4}{9}g_E \). Thus

\[ g_T = \frac{GM_T}{R_T^2} = \frac{GM_T}{9 \times R_E^2} = \frac{4}{9}g_E = \frac{4}{9} \frac{GM_E}{R_E^2}. \]  

(7)

Cancelling common factors, we find that \( M_T = 4M_E \), so Terra is more massive by a factor of 4.

8. (15 pts.) You have a hovercraft whose thrusters keep a gap between the bottom of the hovercraft and the ground.

(a) If the force of the thruster is carefully set to exactly cancel the force of gravity from Terra (and assuming no other forces on the hovercraft), what can you say about the motion of the hovercraft based on Newton’s Laws. Be specific as to which laws you are using by either naming the number of the Law or restating it.

If the force of the thruster exactly cancels gravity, and there are no other forces, then the net force on the hovercraft is zero. By Newton’s second law \( \vec{F} = m \vec{a} \) the hovercraft is not accelerating, so its velocity is constant by Newton’s first law (in the absence of forces, a body in motion maintains its motion.) The key point here is that although there is no force on the hovercraft, it’s wrong to conclude that it is stationary.

(b) Draw a free body diagram for the hovercraft. Be sure to label the figure so it is clear what each part means.

The free body diagram should look something like:

(c) Later, the hovercraft is moving along the Terran surface at a constant speed of 10 m/s. A drawing of its path is
shown below with letters to mark various parts of the path. Its speed is the same over each part of the path. Circle the letters which indicate parts where the net force on the hovercraft is zero (if any). Explain your answer.

The net force on the hovercraft is zero for regions A, C, E. You are told that the speed is constant, but in regions B and D the direction of the velocity is changing, implying an acceleration. The direction of \( \vec{v} \) is constant for A, C, E, as well as the magnitude, so the net force in these regions must be zero.

9. (30 pts.) Recall that the velocity needed to escape the Earth’s gravitational force and travel to deep space is about 11 km/s. You want to send up a reconnaissance satellite so you can track what is going on the rest of the Terran surface. The satellite has a mass of 200 kg. What is the minimal velocity will you need if you want your satellite to escape Terra’s gravity and travel to deep space?

If the satellite is in deep space, far away from all massive objects, its potential energy is zero. Thus conservation of energy implies that to escape from Terra’s gravity, the satellite must have total energy at least zero. The total energy at the planet’s surface is \( E = \frac{1}{2}mv^2 - \frac{GMm}{r} \), where \( M \) and \( R \) are the planet’s mass and radius, so that the condition \( E \geq 0 \) implies \( v \geq \sqrt{\frac{2GM}{R}} \). The escape velocity is the minimum velocity needed to escape: \( v_{esc} = \sqrt{\frac{2GM}{R}} \).

This expression is valid for escape from both Terra and Earth, with appropriate values for \( M \) and \( R \) in each case. Using the values of \( MT \) and \( RT \) found earlier,

\[
v_{esc,T} = \sqrt{\frac{2GM_T}{R_T}} = \sqrt{\frac{4}{3} \times \frac{2GM_E}{R_E}} = \frac{2}{\sqrt{3}} \times v_{esc,E}.
\]

This answer was sufficient for the graders, but if you computed a numerical value, it should have been about 13 km/s.

(b) Your rocket weighs 1000 kg. When it is time to go back to Earth, what is the minimal velocity your engines must give the rocket in order to escape from Terra’s gravity?

Notice from the derivation above that the escape velocity is independent of the mass of the object we are trying to propel into deep space. Therefore the minimal velocity for the rocket to escape Terra’s gravity is the same as that for the satellite. See also the solution to 3(a).

10. (25 pts.) You try sending up the satellite in a rocket. The rocket is of unusual design. You launch it horizontally, and then its thrusters fire directly downwards, producing a force on the rocket that is directly upwards. The thrusters are on for THREE seconds only. If the horizontal velocity was initially 30 km/s, and the acceleration due to thrusters is 20 km/s² upwards, graph the path of the rocket for the first FOUR seconds. Let \( x \) represent the horizontal position and \( y \) the vertical position. Indicate with a small circle on the plot where it is after 1, 2, 3, and 4 seconds. Show all work.

This problem can be solved using the generic equation for motion with constant acceleration: \( r = r_0 + \vec{v}_0t + \frac{1}{2}\vec{a}t^2 \). The things you need to know are that the components of this equation describe the motion in the \( x \) and \( y \) directions separately, and that you can break up the motion into two parts (the first three seconds, when the thrusters are on, and the final second, when the thrusters are off but the rocket maintains its velocity, by Newton’s first law.) The horizontal motion is the simple part – there are no forces in the horizontal direction, so for the entire trajectory \( x = v_0x t = 30t \) km, where \( t \) is measured in seconds. For the first three seconds, the vertical motion is described by
the equation $y = \frac{1}{2}ay^2 = 10t^2$ km. At $t = 3$ s, the thrusters turn off. At this time, $y = 90$ km and $v_y(t = 3) = 60$ km/s. So after the thrusters turn off, the equation for the $y$ coordinate is $y = 90 + 60(t - 3)$. km.

<table>
<thead>
<tr>
<th>Time (s)</th>
<th>x (km)</th>
<th>y(km)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>30</td>
<td>10</td>
</tr>
<tr>
<td>2</td>
<td>60</td>
<td>40</td>
</tr>
<tr>
<td>3</td>
<td>90</td>
<td>90</td>
</tr>
<tr>
<td>4</td>
<td>120</td>
<td>150</td>
</tr>
</tbody>
</table>

Your graph should have looked something like this:
11. (20 pts.) The satellite travels upwards into space, fires some rockets, and comes to a halt. Just at that moment, it explodes. It breaks into three chunks. The first, 100 kg, flies off directly downward at 5 km/s. The second, 50 kg, flies out up and to the right by 60 degrees. See the figure. The third chunk is 50 kg, too. Which way does it go flying and with what speed. Show work and indicate your result by completing the sketch below.

![Diagram](image)

Note that the diagram showed that the second mass has a speed of 10 km/s. By momentum conservation, the third mass (50 kg) must fly off with speed 10 km/s in the direction 60 degrees to the left of the vertical direction, as shown below. The figure shows the velocities of the masses, but the really important quantity is momentum, \( \vec{p} = m \vec{v} \). It is easy to check that the horizontal components of momentum add up to zero. To check the vertical component, remember that the short leg of a 30-60-90 triangle is half as long as the hypotenuse. So the two masses moving to the left and right each contribute \( 50 \times 10 \times \frac{1}{2} \) to the vertical momentum, so their sum cancels the \(-500 \) km-kg/s from the first mass.

12. (15 pts.) After you dust yourself off from that disaster, you notice that Terra has three moons. Luna-1, Luna-2, and Luna-3. You compile some measurements of the period and distance.

<table>
<thead>
<tr>
<th>Moon</th>
<th>Period (months)</th>
<th>Semi-major axis (Terra units)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Luna-1</td>
<td>9</td>
<td>1</td>
</tr>
<tr>
<td>Luna-2</td>
<td>?</td>
<td>4</td>
</tr>
<tr>
<td>Luna-3</td>
<td>243</td>
<td>?</td>
</tr>
</tbody>
</table>

Determine the period of Luna-2 and the semi-major axis of Luna-3. Show work.

Recall one of Kepler’s laws, \( \frac{P^2}{a^3} = \text{constant} \) for the orbits of objects about the same (large) central body. To determine the period of Luna-2, note that in the given units,

\[
\frac{P_2^2}{a_2^3} = 9^2 = \frac{P_2^2}{a_2^3} = \frac{P_2^2}{4^3} \Rightarrow P_2 = \sqrt{9^2 \times 4^3} = 9 \times 2^3 = 72. \tag{9}
\]

So the period of Luna-2 is 72 months. To get the semi-major axis of Luna-3, you play the same game but solve for \( a \) instead of \( P \) this time.
\[
\frac{P_1^2}{a_1^3} = 9^2 = \frac{P_2^2}{a_2^3} = \frac{243^2}{a_3^3} \Rightarrow a_3 = \sqrt[3]{\frac{243^2}{9^2}} = \sqrt[3]{\frac{3^{10}}{3^4}} = \sqrt[3]{3^6} = 3^2 = 9.
\] (10)

Thus the length of semi-major axis of Luna-3’s orbit is 9 Terra units.

13. (15 pts.) The orbits of Luna-1 and Luna-2 are circular. Is Luna-1 or Luna-2 traveling at the faster speed? By what factor? Show work.

For a circular orbit, the speed is constant, with \( |\vec{v}| = \frac{2\pi a}{P} \). From Kepler’s law \( \frac{P^2}{a^3} = \text{constant} \), you can show that \( v^2 = \frac{k'}{a} \), where \( k' \) is another constant. Taking ratios to cancel \( k' \), we find that

\[
\frac{v_1^2}{v_2^2} = \frac{a_2}{a_1} = 4 \Rightarrow v_1^2 = 4 \times v_2^2 \Rightarrow v_1 = 2 \times v_2.
\] (11)

As we have shown, Luna-1 is moving faster than Luna-2, by a factor of 2.

14. (15 pts.) The orbit of Luna-3 is far from circular.

(a) Sketch the orbits of the Moons around Terra as carefully as possible.

A common mistake was to draw Terra at the center of the orbit of Luna-3, instead of at the focus of the ellipse. Moreover, Terra, the point of closest approach of Luna-3, and the point at which Luna-3 is most distant from Terra, should all lie on the same line.

(b) On your diagram above, draw two arrows indicating the velocity and acceleration of Luna-1 at two opposite points in its orbit.

Notice that the velocity vector always points towards along the direction of motion, but the acceleration vector points from the moon to the planet.

(c) Describe in words the orbit of Luna-3.
We are told that the orbit is not circular, so in the Keplerian approximation (that is, we treat Terra as stationary) the orbit is an ellipse (basically a stretched circle.) Terra is at one of the foci of the ellipse. At the point where Luna-3 is closest to Terra, its speed is at its greatest value, and at the farthest point, its speed is at its smallest value (this follows from Kepler’s law of equal areas in equal times.)