

“Hidden” Momentum in an Oscillating Spring

Kirk T. McDonald

Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544

(July 8, 2012; updated August 18, 2012)

1 Problem

The term “hidden” momentum was popularized by Shockley [1] in considerations of an electromechanical example, and essentially all subsequent use of this term has been for such examples, where one considers the system to consist of matter plus electromagnetic fields.

Recently, a definition of “hidden” momentum has been proposed by Daniel Vanzella [2] (see also [3]) which can be applied to mechanical systems as well, where a subsystem has a specified volume and can interact with the rest of the system via contact forces and/or transfer of mass/energy across its surface (which can be in motion),

$$\mathbf{P}_{\text{hidden}} \equiv \mathbf{P} - M\mathbf{v}_{\text{cm}} - \oint_{\text{boundary}} (\mathbf{x} - \mathbf{x}_{\text{cm}}) (\mathbf{p} - \rho\mathbf{v}_b) \cdot d\mathbf{Area} = - \int \frac{f^0}{c} (\mathbf{x} - \mathbf{x}_{\text{cm}}) d\text{Vol}, \quad (1)$$

where \mathbf{P} is the total momentum of the subsystem, $M = U/c^2$ is its total “mass”, U is its total energy, c is the speed of light in vacuum, \mathbf{x}_{cm} is its center of mass/energy, $\mathbf{v}_{\text{cm}} = d\mathbf{x}_{\text{cm}}/dt$, \mathbf{p} is its momentum density, $\rho = u/c^2$ is its “mass” density, u is its energy density, \mathbf{v}_b is the velocity (field) of its boundary, and

$$f^\mu = \frac{\partial T^{\mu\nu}}{\partial x^\nu}, \quad (2)$$

is the 4-force density exerted on the subsystem by the rest of the system, with $T^{\mu\nu}$ being the stress-energy-momentum 4-tensor of the subsystem.

Does an isolated, oscillating spring contain “hidden” momentum according to the above definition?

2 Solution

Aug. 18, 2012. This answer is NO, but I had made a computational error such that I thought the answer was YES. This error is only partially fixed below.

We consider the spring to be a bar of rest mass m , rest length L , cross sectional area A and Young’s (elastic) modulus E . For simplicity, we assume that Poisson’s ratio is zero for the bar/spring. The center of the bar is at rest at the origin, and the oscillations are in the x -coordinate.

2.1 Nonrelativistic Oscillation

The equation of motion for the displacement $s(x, t)$ of an element dx of the cable, centered on x , is then

$$m \frac{dx}{L} \ddot{s} = F(x + dx) - F(x) = F'(x) dx, \quad (3)$$

where $F(x)$ is the internal force across a y - z plane through point $(x, 0, 0)$. That is, t

$$\ddot{s} = \frac{L}{m} F'(x). \quad (4)$$

Due to the internal force f the element dx has stretched by amount

$$\Delta x = s(x + dx) - s(x) = s'(x)dx. \quad (5)$$

We recall that stretching of an elastic medium can be related to its elastic modulus E via

$$\frac{F}{A} = E \frac{\Delta L}{L}. \quad (6)$$

Considering the entire bar/spring, the modulus E and the spring constant k are related by

$$K = \frac{EA}{L}. \quad (7)$$

For the element dx , eq. (6) becomes

$$F = EA \frac{\Delta x}{dx} = \frac{KL}{dx} \Delta x = KLs'. \quad (8)$$

Inserting this in the equation of motion (4) we find the wave equation

$$\ddot{s} = \frac{KL^2}{m} s''. \quad (9)$$

We seek standing wave solutions of angular frequency ω :

$$s = g(x) \cos \omega t. \quad (10)$$

Inserting this in eq. (9) we find

$$g'' = -\frac{m\omega^2}{KL^2} g, \quad (11)$$

which is solved by

$$g = a \sin \sqrt{\frac{m}{K} \frac{\omega x}{L}}, \quad (12)$$

noting that the center of mass must remain fixed at $x = 0$, which requires the displacements to be antisymmetric in x .

The boundary conditions at the ends of the bar, $x = \pm L/2$, are that the stretch is zero there,

$$s'(\pm L/2, t) = 0, \quad (13)$$

which implies that

$$\sqrt{\frac{m}{K} \frac{\omega}{2}} = \frac{(2n+1)\pi}{2}, \quad (14)$$

for n an integer. We will consider only the lowest mode of oscillation, $n = 0$, for which the angular frequency of oscillation is

$$\omega = \pi \sqrt{\frac{K}{m}}, \quad (15)$$

and the standing waveform is

$$s(x, t) = a \sin \frac{\pi x}{L} \cos \omega t. \quad (16)$$

The internal force F of eq. (8) is

$$F(x, t) = KLs' = \pi a K \cos \frac{\pi x}{L} \cos \omega t, \quad (17)$$

where positive f implies that the element dx is under tension. The velocity \dot{s} is

$$\dot{s}(x, t) = -a\omega \sin \frac{\pi x}{L} \sin \omega t, \quad (18)$$

2.1.1 Stress-Energy-Momentum Tensor

In the rest frame (the \star frame) of an element dx of the bar/spring the stress-energy-momentum tensor has the form

$$T^{\star\mu\nu} = \left(\begin{array}{c|ccc} \rho^\star c^2 & & & \mathbf{0} \\ \hline & -F/A & 0 & 0 \\ \mathbf{0} & 0 & 0 & 0 \\ & 0 & 0 & 0 \end{array} \right), \quad (19)$$

where the effective mass density ρ^\star is normalized to the unstretched volume $A dx$, and includes a contribution from the elastic energy of the stretched segment (whose spring constant is KL/dx , recalling eq. (8)),

$$\rho^\star = \frac{m}{AL} + \frac{1}{c^2} \frac{1}{A} \frac{1}{dx} \frac{1}{2} \frac{KL}{dx} \Delta x^2 = \frac{m}{AL} + \frac{KLs'^2}{2Ac^2}, \quad (20)$$

recalling eq. (5).

In the lab frame the segment has velocity \dot{s} with Lorentz factor

$$\gamma(x) = \frac{1}{\sqrt{1 - \dot{s}^2/c^2}}, \quad (21)$$

and the Lorentz transformation from the \star frame of the segment to the lab frame is

$$\mathbf{L}^{\mu\nu}(x) = \left(\begin{array}{c|ccc} \gamma & \gamma\dot{s}/c & 0 & 0 \\ \hline \gamma\dot{s}/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right). \quad (22)$$

Hence, the stress-energy-momentum tensor in the lab frame is

$$T^{\mu\nu} = (\text{LT}^*\text{L})^{\mu\nu} = \left(\begin{array}{c|cc} \gamma^2(\rho^*c^2 - \dot{s}^2F/Ac^2) & \gamma^2\dot{s}(\rho^*c^2 - F/A)/c & 0 & 0 \\ \hline \gamma^2\dot{s}(\rho^* - F/A)/c & \gamma^2(\dot{s}^2\rho^* - F/A) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right). \quad (23)$$

2.1.2 “Hidden” Momentum

The momentum density \mathbf{p} in the oscillating bar/spring is

$$\begin{aligned} \mathbf{p} &= \frac{T^{0x}}{c} \hat{\mathbf{x}} = \gamma^2\dot{s} \left(\rho^* - \frac{F}{Ac^2} \right) \hat{\mathbf{x}} \approx \left(1 + \frac{\dot{s}^2}{c^2} \right) \dot{s} \left(\frac{m}{AL} + \frac{KLs'^2}{2Ac^2} - \frac{KLs'}{Ac^2} \right) \hat{\mathbf{x}} \\ &\approx \dot{s} \left[\frac{m}{AL} \left(1 + \frac{\dot{s}^2}{c^2} \right) + \frac{KLs'^2}{2Ac^2} - \frac{KLs'}{Ac^2} \right] \hat{\mathbf{x}} \end{aligned} \quad (24)$$

and the total momentum \mathbf{P} is

$$\mathbf{P} = A \int_{-L/2}^{L/2} \mathbf{p} dx = 0, \quad (25)$$

as \dot{s} is antisymmetric in x while s' is symmetric. Of course, the position \mathbf{x}_{cm} and the velocity \mathbf{v}_{cm} of the center of mass/energy are zero in the lab frame. If we take the boundary of the bar/spring to be its physical surface, then the velocity of the bounding surfaces corresponding to $x = \pm L/2$ is $\mathbf{v}_b = \dot{s} \hat{\mathbf{x}}$. Then, the boundary integral in the first form of “hidden” momentum in eq. (1) is

$$\begin{aligned} &\oint_{\text{boundary}} (\mathbf{x} - \mathbf{x}_{\text{cm}}) (\mathbf{p} - \rho \mathbf{v}_b) \cdot d\text{Area} \\ &= \frac{AL}{2} [p_x(L/2) + p_x(-L/2) - \rho(L/2)\dot{s}(L/2) - \rho(-L/2)\dot{s}(-L/2)] \hat{\mathbf{x}} = 0, \end{aligned} \quad (26)$$

and hence the “hidden” momentum is zero,

$$\mathbf{P}_{\text{hidden}} = \mathbf{P} - M\mathbf{v}_{\text{cm}} - \oint_{\text{boundary}} (\mathbf{x} - \mathbf{x}_{\text{cm}}) (\mathbf{p} - \rho \mathbf{v}_b) \cdot d\text{Area} = 0, \quad (27)$$

in the approximation of the nonrelativistic form (18) of the oscillation.

Consider now the subsystem consisting of the bar/spring at $x > 0$.

The total momentum \mathbf{P} is

$$\mathbf{P}(x > 0) = A \int_0^{L/2} \mathbf{p} dx \approx \frac{m}{L} \int_0^{L/2} \dot{s} \left(1 + \frac{\dot{s}^2}{c^2} + \frac{KL^2s'^2}{2mc^2} - \frac{KL^2s'}{mc^2} \right) dx \hat{\mathbf{x}}. \quad (28)$$

The mass density ρ is

$$\rho = \frac{T^{00}}{c^2} \approx \gamma^2\rho^* \approx \frac{m}{AL} \left(1 + \frac{\dot{s}^2}{c^2} \right) + \frac{KLs'^2}{2Ac^2} = \frac{m}{AL} \left(1 + \frac{\dot{s}^2}{c^2} + \frac{KL^2s'^2}{2mc^2} \right), \quad (29)$$

so

$$M = A \int_0^{L/2} \rho dx \approx \frac{m}{L} \int_0^{L/2} \left(1 + \frac{\dot{s}^2}{c^2} + \frac{KL^2 s'^2}{2mc^2} \right) dx, \quad (30)$$

and

$$\dot{M} \approx \frac{m}{L} \int_0^{L/2} \left(\frac{2\dot{s}\ddot{s}}{c^2} + \frac{KL^2 s' \dot{s}'}{mc^2} \right) dx. \quad (31)$$

We can also calculate

$$Mx_{\text{cm}} = A \int_0^{L/2} (x+s)\rho dx \approx \frac{m}{L} \int_0^{L/2} (x+s) \left(1 + \frac{\dot{s}^2}{c^2} + \frac{KL^2 s'^2}{2mc^2} \right) dx. \quad (32)$$

Then,

$$M\dot{x}_{\text{cm}} = \frac{d(Mx_{\text{cm}})}{dt} - \dot{M}Mx_{\text{cm}}, \quad (33)$$

where from eq. (32),

$$\frac{d(Mx_{\text{cm}})}{dt} \approx \frac{m}{L} \int_0^{L/2} \dot{s} \left(1 + \frac{\dot{s}^2}{c^2} + \frac{KL^2 s'^2}{2mc^2} \right) dx + \frac{m}{L} \int_0^{L/2} (x+s) \left(\frac{2\dot{s}\ddot{s}}{c^2} + \frac{KL^2 s' \dot{s}'}{mc^2} \right), \quad (34)$$

and the boundary integral is

$$\begin{aligned} & \oint_{\text{boundary}} (\mathbf{x} - \mathbf{x}_{\text{cm}}) (\mathbf{p} - \rho \mathbf{v}_b) \cdot d\text{Area} \\ &= \frac{AL}{2} [p_x(L/2) - \rho(L/2)\dot{s}(L/2) - p_x(0) - \rho(0)\dot{s}(0)] \hat{\mathbf{x}} \\ &\approx \frac{AL}{2} \left[\frac{m}{AL} \dot{s} \left(1 + \frac{\dot{s}^2}{c^2} + \frac{KL^2 s'^2}{2mc^2} - \frac{KL^2 s'}{mc^2} \right) - \frac{m}{AL} \dot{s} \left(1 + \frac{\dot{s}^2}{c^2} + \frac{KL^2 s'^2}{2mc^2} \right) \right]_{L/2} \hat{\mathbf{x}} \\ &= -\frac{KL^2 \dot{s} s'}{2c^2} \Big|_{L/2} \hat{\mathbf{x}} = 0. \end{aligned} \quad (35)$$

Combining these results, the x -component of the “hidden” momentum in the bar/spring for $x > 0$ is

$$\begin{aligned} P_{x,\text{hidden}}(x > 0) &= P_x - M\dot{x}_{\text{cm}} - \oint_{\text{boundary}} (x - x_{\text{cm}}) (\mathbf{p} - \rho \mathbf{v}_b) \cdot d\text{Area} \\ &\approx \frac{m}{L} \int_0^{L/2} -\dot{s} \frac{KL^2 s'}{mc^2} dx - \frac{m}{L} \int_0^{L/2} (x+s) \left(\frac{2\dot{s}\ddot{s}}{c^2} + \frac{KL^2 s' \dot{s}'}{mc^2} \right) + \dot{M}Mx_{\text{cm}}. \end{aligned} \quad (36)$$

The first integral is zero since $\dot{s} s' \propto \sin 2\pi x/L$. In the second integral,

$$\frac{2\dot{s}\ddot{s}}{c^2} + \frac{KL^2 s' \dot{s}'}{mc^2} = \frac{a^2 \omega^3 \sin 2\omega t}{2c^2} \left(3 \sin^2 \frac{\pi x}{L} - 1 \right), \quad (37)$$

so, noting that

$$\int_0^{L/2} x \left(3 \sin^2 \frac{\pi x}{L} - 1 \right) dx = \frac{L^2}{16}, \quad \int_0^{L/2} \sin \frac{\pi x}{L} \left(3 \sin^2 \frac{\pi x}{L} - 1 \right) dx = \frac{L}{\pi}, \quad (38)$$

we have

$$P_{x,\text{hidden}}(x > 0) \approx -\frac{ma^2\omega^3 \sin 2\omega t}{2c^2} \left(\frac{L}{16} + \frac{a \cos \omega t}{\pi} \right) + \frac{\dot{M}}{M} M x_{\text{cm}}. \quad (39)$$

I am now confident that eq. (39) is zero, if properly computed, although there seemed to a factor of 2 loose in one term when I tried it.

Possibly the “hidden” momentum would be zero at order $1/c^2$ if we used the relativistic form of the oscillation.

2.2 Relativistic Oscillation at Order $1/c^2$

“Hidden” momentum in electromechanical examples is always of order $1/c^2$, so we consider the oscillations of the bar/spring at this order.

The relativistic version of the equation of motion (9) is

$$\gamma K L^2 s'' = \frac{d}{dt} \gamma m \dot{s} = \dot{\gamma} m \dot{s} + \gamma m \ddot{s} = \gamma m \ddot{s} \left(1 + \frac{2\gamma^2 \dot{s}^2}{c^2} \right) \approx m \ddot{s} \left(1 + \frac{5\dot{s}^2}{2c^2} \right), \quad (40)$$

where

$$\gamma = \frac{1}{\sqrt{1 - \dot{s}^2/c^2}}, \quad (41)$$

and the approximation holds at order $1/c^2$. We seek a perturbative solution of the form

$$s = s_0 + s_1 + \dots, \quad (42)$$

where s_0 satisfies the nonrelativistic equation of motion (9) and is given by eq. (16). Then, s_1 is of order $1/c^2$, obeys the boundary condition that $s'_1(\pm L/2) = 0$, and vanishes at $x = 0$ (being a term in the lowest mode of oscillation). Using eq. (42) in eq. (36) for the “hidden” momentum at $x > 0$, the first integral is again zero, while the terms in the second integral due to s_1 are of order $1/c^4$.

Thus, the nonzero “hidden” momentum (39) remains valid at order $1/c^2$, and provides an example of “hidden” momentum in a subsystem of an all-mechanical system.

References

- [1] W. Shockley and R.P. James, “Try Simplest Cases” Discovery of “Hidden Momentum” Forces on “Magnetic Currents”, *Phys. Rev. Lett.* **18**, 876 (1967), http://physics.princeton.edu/~mcdonald/examples/EM/shockley_prl_18_876_67.pdf
- [2] D. Vanzella, *Hidden momentum of (possibly open) systems* (June 29, 2012), http://physics.princeton.edu/~mcdonald/examples/EM/vanzella_120629.pdf
- [3] K.T. McDonald, *On the Definition of “Hidden” Momentum* (July 9, 2012), <http://physics.princeton.edu/~mcdonald/examples/hiddendef.pdf>