

# Orbital and Spin Angular Momentum of Electromagnetic Fields

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## 1 Problem

Poynting [1] identified the flux of energy in the electromagnetic fields  $\{\mathbf{E}, \mathbf{B}\}$  (in a medium with relative permittivity  $\epsilon = 1$  and relative permeability  $\mu = 1$ ) with the vector,

$$\mathbf{S} = \frac{c}{4\pi} \mathbf{E} \times \mathbf{B}, \quad (1)$$

in Gaussian units, where  $c$  is the speed of light in vacuum. Abraham [2] recognized the additional role of the Poynting vector as being proportional to the density of linear momentum stored in the electromagnetic field,

$$\mathbf{p}_{\text{EM}} = \frac{\mathbf{S}}{c^2} = \frac{\mathbf{E} \times \mathbf{B}}{4\pi c}, \quad (2)$$

although these arguments most clearly show that the volume integral,

$$\mathbf{P}_{\text{EM}} = \int \mathbf{p}_{\text{EM}} d\text{Vol}, \quad (3)$$

rather than the integrand (2), has physical significance. This suggests that the density of angular momentum stored in the electromagnetic field can be written as,

$$\mathbf{l}_{\text{EM}} = \mathbf{r} \times \mathbf{p}_{\text{EM}} = \mathbf{r} \times \frac{\mathbf{E} \times \mathbf{B}}{4\pi c}. \quad (4)$$

Show that the Helmholtz decomposition [3] of a vector field  $\mathbf{F}$  into irrotational and rotational parts,

$$\mathbf{F} = \mathbf{F}_{\text{irr}} + \mathbf{F}_{\text{rot}}, \quad (5)$$

where,

$$\nabla \times \mathbf{F}_{\text{irr}} = 0, \quad \text{and} \quad \nabla \cdot \mathbf{F}_{\text{rot}} = 0 \quad (6)$$

at **all** points in space, leads to alternative forms for the densities of momentum and angular momentum in the electromagnetic fields of a system of charges  $e_i$  of rest masses  $m_i$  and velocities  $\mathbf{v}_i$ :

$$\mathbf{P}_{\text{total}} = \sum_i \mathbf{p}_{\text{canonical}, i} + \int \mathbf{p}_{\text{EM}, \text{orbital}} d\text{Vol}, \quad (7)$$

and,

$$\mathbf{L}_{\text{total}} = \sum_i \mathbf{l}_{\text{canonical}, i} + \int \mathbf{l}_{\text{EM}, \text{orbital}} d\text{Vol} + \int \mathbf{l}_{\text{EM}, \text{spin}} d\text{Vol}, \quad (8)$$

where,

$$\mathbf{p}_{\text{canonical},i} = \gamma_i m \mathbf{v}_i + \mathbf{p}_{\text{EM,canonical},i}, \quad \gamma_i = \frac{1}{\sqrt{1 - v_i^2/c^2}}, \quad \mathbf{p}_{\text{EM,canonical},i} = \frac{e_i \mathbf{A}_{\text{rot}}(\mathbf{r}_i)}{c}, \quad (9)$$

$$\mathbf{p}_{\text{EM,orbital}} = \frac{\sum_{j=1}^3 E_{\text{rot},j} \nabla A_{\text{rot},j}}{4\pi c}, \quad (10)$$

and,

$$\mathbf{l}_{\text{canonical},i} = \mathbf{r} \times \mathbf{p}_{\text{canonical},i}, \quad \mathbf{l}_{\text{EM,orbital}} = \mathbf{r} \times \mathbf{p}_{\text{EM,orbital}}, \quad \mathbf{l}_{\text{EM,spin}} = \frac{\mathbf{E}_{\text{rot}} \times \mathbf{A}_{\text{rot}}}{4\pi c}, \quad (11)$$

where  $\mathbf{A}_{\text{rot}}$  is the rotational part of the (gauge-dependent) vector potential  $\mathbf{A}$ , and  $\mathbf{A}_{\text{rot}}(\mathbf{r}_i)$  is the rotational part of the vector potential at charge  $i$  due to all other sources.<sup>1,2</sup>

## 2 Solution

The identification of orbital and spin parts of the electromagnetic angular momentum was anticipated by Henriot [5], but may be due to Rosenfeld [6, 7], who examined “classical” field theories of particles of various spin. For the latter, see also [8, 9]. The flux of orbital and spin angular momentum was considered by Humblet [10]. These considerations are little represented in treatises on classical electrodynamics, but they are well summarized in chap. 1 of [11], which this solution largely follows. See also chap. 9 of [12], and [13]. Alternative decompositions are discussed, for example, in [14].<sup>3</sup>

### 2.1 Helmholtz Decomposition of the Electromagnetic Fields and the Coulomb Gauge

Helmholtz [3, 4] showed (in a hydrodynamic context) that any vector field, say  $\mathbf{F}$ , which vanishes suitably quickly at infinity can be decomposed according to eq. (5), where the irrotational and rotational (or solenoidal)<sup>4</sup> components  $\mathbf{F}_{\text{irr}}$  and  $\mathbf{F}_{\text{rot}}$  obey eq. (6).

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<sup>1</sup>It is claimed in eq. (B.26), p. 17, of [11] that  $\mathbf{A}_{\text{rot}}$  (called  $\mathbf{A}_{\perp}$  there) is gauge invariant, but this is not so, as discussed in Appendix B below. See also Appendix B.1 of [4].

<sup>2</sup>The rotational electric field,  $\mathbf{E}_{\text{rot}}$ , is nonzero only in time-dependent situations, according to eq. (25), so the quantities  $\mathbf{p}_{\text{EM,orbital}}$ ,  $\mathbf{l}_{\text{EM,orbital}}$  and  $\mathbf{l}_{\text{EM,spin}}$  are nonzero only in such cases.

<sup>3</sup>These considerations were largely unknown to the elementary-particle-physics community until rediscovered in 2007, [15], after which a flurry of papers has ensued, as reviewed in [16].

<sup>4</sup>The irrotational and rotational/solenoidal components  $\mathbf{F}_{\text{irr}}$  and  $\mathbf{F}_{\text{rot}}$  are called the longitudinal and transverse components,  $\mathbf{F}_{\parallel}$  and  $\mathbf{F}_{\perp}$  respectively, by some people. The latter nomenclature derives from plane waves,  $\mathbf{F} = \mathbf{F}_0 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$ , to which the proof of Helmholtz decomposition does not formally apply, but which is readily written as  $\mathbf{F}_{\text{irr}} = \mathbf{F}_{\parallel} = (\mathbf{F} \cdot \hat{\mathbf{k}}) \hat{\mathbf{k}}$  and  $\mathbf{F}_{\text{rot}} = \mathbf{F}_{\perp} = \mathbf{F} - \mathbf{F}_{\parallel}$  such that  $\mathbf{F}_{\perp} \cdot \mathbf{k} = 0$ , and the irrotational/longitudinal and rotational/solenoidal/transverse components of  $\mathbf{F}$  are parallel and perpendicular, respectively, to the wave vector  $\mathbf{k}$ . However,  $\hat{\mathbf{k}}$  is not defined for  $\mathbf{k} = 0$ , and neglect of this can lead to misunderstandings, such as that mentioned in footnote 1.

The author prefers the terms irrotational and rotational to describe the global argument of Helmholtz, because the terms longitudinal and transverse fields commonly describe only local aspects of vector fields.

Helmholtz also showed that,<sup>5</sup>

$$\mathbf{F}_{\text{irr}}(\mathbf{r}) = -\nabla \int \frac{\nabla' \cdot \mathbf{F}(\mathbf{r}')}{4\pi R} d\text{Vol}', \quad \text{and} \quad \mathbf{F}_{\text{rot}}(\mathbf{r}) = \nabla \times \int \frac{\nabla' \times \mathbf{F}(\mathbf{r}')}{4\pi R} d\text{Vol}', \quad (13)$$

where  $R = |\mathbf{r} - \mathbf{r}'|$ . Time does not appear in eq. (13), which indicates that the vector field  $\mathbf{F}$  at some point  $\mathbf{r}$  (and some time  $t$ ) can be reconstructed from knowledge of its vector derivatives,  $\nabla \cdot \mathbf{F}$  and  $\nabla \times \mathbf{F}$ , over all space (at the same time  $t$ ).

An important historical significance of the Helmholtz decomposition (5) and (12) was in showing that Maxwell's equations, which give prescriptions for the derivatives of the electromagnetic fields  $\mathbf{E}$  and  $\mathbf{B}$ , are mathematically sufficient to determine those fields.

In this note we consider only media with relative permittivity  $\epsilon = 1$  and relative permeability  $\mu = 1$ , so that Maxwell's equations can be written (in Gaussian units) in terms of the macroscopic charge and current densities,  $\rho$  and  $\mathbf{J}$ , as,

$$\nabla \cdot \mathbf{E} = 4\pi\rho, \quad (14)$$

$$\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}, \quad (15)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (16)$$

$$\nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{J} + \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t}, \quad (17)$$

where  $c$  is the speed of light in vacuum.

It follows from eq. (16) that the magnetic field  $\mathbf{B}$  is purely rotational in the sense of Helmholtz,

$$\mathbf{B}_{\text{rot}} = \mathbf{B}. \quad (18)$$

In general, the electric field  $\mathbf{E}$  has both irrotational and rotational components. In Appendix A it is shown that the irrotational part of  $\mathbf{E}$  at time  $t$  is the static (Coulomb) field that would exist if the charge density  $\rho(\mathbf{r}, t)$  had been unchanged for all earlier times,

$$\mathbf{E}_{\text{irr}}(\mathbf{r}, t) = \mathbf{E}^{(C)} = \int \frac{\rho(\mathbf{r}', t) \hat{\mathbf{R}}}{R^2} d\text{Vol}' = \sum_i e_i \frac{\hat{\mathbf{R}}_i}{R_i^2}, \quad (19)$$

where  $\mathbf{R} = \mathbf{r} - \mathbf{r}'$ , and in the microscopic view,  $e_i$  is the electric charge of particle  $i$ . Thus, the electric field can be purely rotational only if the macroscopic charge density  $\rho$  is everywhere zero; in the microscopic view  $\mathbf{E}_{\text{irr}} = 0$  only if all particles are electrically neutral.

That the irrotational part of the electric field can be calculated from the instantaneous charge distribution cautions us that the Helmholtz decomposition (5) does not imply independent physical significance for the partial fields  $\mathbf{E}_{\text{irr}}$  and  $\mathbf{E}_{\text{rot}}$ . In general, only the total electric field  $\mathbf{E}$  has the physical significance of propagation at the speed of light.

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<sup>5</sup>Using the identity that  $(\nabla' \times \mathbf{F}(\mathbf{r}'))/R = \nabla' \times (\mathbf{F}/R) + \mathbf{F} \times \nabla'(1/R) = \nabla' \times (\mathbf{F}/R) + \nabla(1/R) \times \mathbf{F}$ , we can also write,

$$\mathbf{F}_{\text{rot}}(\mathbf{r}) = \nabla \times \int \nabla' \times \frac{\mathbf{F}(\mathbf{r}')}{4\pi R} d\text{Vol}' + \nabla \times \nabla \times \int \frac{\mathbf{F}(\mathbf{r}')}{4\pi R} d\text{Vol}' = \nabla \times \nabla \times \int \frac{\mathbf{F}(\mathbf{r}')}{4\pi R} d\text{Vol}', \quad (12)$$

for fields  $\mathbf{F}$  that vanish quickly enough at infinity.

An explicit expression for the rotational part of the electric field can be given in the Darwin approximation (Appendix C), in which electrodynamics is considered only to order  $1/c^2$ ,

$$\mathbf{E}_{\text{rot}} = - \sum_i \frac{e_i}{2c^2 R_i} \left[ \mathbf{a}_i + (\mathbf{a}_i \cdot \hat{\mathbf{R}}_i) \hat{\mathbf{R}}_i + \frac{3(\mathbf{v}_i \cdot \hat{\mathbf{R}}_i)^2 - v_i^2}{R_i} \hat{\mathbf{R}}_i \right], \quad (20)$$

where  $\mathbf{a}_i$  and  $\mathbf{v}_i$  are the acceleration and velocity of particle  $i$ .

The electric field  $\mathbf{E}$  and the magnetic field  $\mathbf{B}$  can be related to a scalar potential  $V$  and a vector potential  $\mathbf{A}$  according to,

$$\mathbf{E} = -\nabla V - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}, \quad (21)$$

$$\mathbf{B} = \nabla \times \mathbf{A}. \quad (22)$$

The vector field  $-\nabla V$  is purely longitudinal, but in general the vector potential  $\mathbf{A}$  has both longitudinal and transverse components.

The potentials  $V$  and  $\mathbf{A}$  are not unique, but can be redefined in a systematic way such that the fields  $\mathbf{E}$  and  $\mathbf{B}$  are invariant under such redefinition. A particular choice of the potentials is called a choice of *gauge*,<sup>6</sup> and the relations (16)-(17) are said to be *gauge invariant*. The gauge transformation,

$$\mathbf{A} \rightarrow \mathbf{A} + \nabla \chi, \quad V \rightarrow V - \frac{1}{c} \frac{\partial \chi}{\partial t}, \quad (23)$$

leaves the fields  $\mathbf{E}$  and  $\mathbf{B}$  unchanged.

If we work in the Coulomb gauge (see Appendix B), where  $\nabla \cdot \mathbf{A}^{(C)} = 0$ , then  $\mathbf{A}_{\text{irr}}^{(C)} = 0$  and  $\mathbf{A}_{\text{rot}}^{(C)} = \mathbf{A}^{(C)} = \mathbf{A}_{\text{rot}}^{(C)}$ , so that,

$$\mathbf{E} = -\nabla V^{(C)} - \frac{\partial \mathbf{A}^{(C)}}{\partial t} = -\nabla V^{(C)} - \frac{\partial \mathbf{A}_{\text{rot}}^{(C)}}{\partial t} = \mathbf{E}_{\text{irr}} + \mathbf{E}_{\text{rot}}, \quad (24)$$

where,

$$\mathbf{E}_{\text{irr}} = -\nabla V^{(C)}, \quad \mathbf{E}_{\text{rot}} = -\frac{1}{c} \frac{\partial \mathbf{A}^{(C)}}{\partial t} = -\frac{1}{c} \frac{\partial \mathbf{A}_{\text{rot}}^{(C)}}{\partial t}. \quad (25)$$

If we work in some other gauge with potentials  $\mathbf{A}$  and  $V$  where the vector potential has both irrotational and rotational parts,  $\mathbf{A} = \mathbf{A}_{\text{irr}} + \mathbf{A}_{\text{rot}}$ , then the decomposition of the electric field is,<sup>7</sup>

$$\mathbf{E}_{\text{irr}} = -\nabla V - \frac{1}{c} \frac{\partial \mathbf{A}_{\text{irr}}}{\partial t}, \quad \mathbf{E}_{\text{rot}} = -\frac{1}{c} \frac{\partial \mathbf{A}_{\text{rot}}}{\partial t}. \quad (26)$$

The decomposition (24)-(26) of the electric field  $\mathbf{E}$  into irrotational and rotational fields  $\mathbf{E}_{\text{irr}}$  and  $\mathbf{E}_{\text{rot}}$  is gauge invariant, but the simplicity of eq. (25) gives a special importance to the Coulomb gauge. However, one must remain cautious about assigning a direct physical significance to  $\mathbf{A}_{\text{rot}}$  because it leads to the field  $\mathbf{E}_{\text{rot}}$  which has components that propagate instantaneously.<sup>8</sup>

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<sup>6</sup>More precisely, a particular set of potentials is said to satisfy a *gauge condition* on  $\nabla \cdot \mathbf{A}$ , and in general there exists a restricted set of gauge-transformation functions  $\chi$  that lead to alternative potentials which also satisfy the gauge condition.

<sup>7</sup>Equations (25)-(26) imply that a general  $\mathbf{A}_{\text{rot}}$  can differ from  $\mathbf{A}_{\text{rot}}^{(C)}$  by a time-independent vector field.

<sup>8</sup>See, for example, [17].

## 2.2 Total Energy of an Electromagnetic System

The electromagnetic energy  $U_{\text{EM}}$  of a system of charges can be written,

$$U_{\text{EM}} = \int \frac{E^2 + B^2}{8\pi} d\text{Vol}. \quad (27)$$

Using the Parseval-Plancherel identity (84), we can write the electric part of the field energy as,

$$\begin{aligned} U_{\text{E}} &= \int \frac{\mathbf{E} \cdot \mathbf{E}}{8\pi} d^3\mathbf{r} = \int \frac{\tilde{\mathbf{E}}^* \cdot \tilde{\mathbf{E}}}{8\pi} d^3\mathbf{k} = \int \frac{(\tilde{\mathbf{E}}_{\text{irr}}^* + \tilde{\mathbf{E}}_{\text{rot}}^*) \cdot (\tilde{\mathbf{E}}_{\text{irr}} + \tilde{\mathbf{E}}_{\text{rot}})}{8\pi} d^3\mathbf{k} \\ &= \int \frac{\tilde{\mathbf{E}}_{\text{irr}}^* \cdot \tilde{\mathbf{E}}_{\text{irr}} + \tilde{\mathbf{E}}_{\text{rot}}^* \cdot \tilde{\mathbf{E}}_{\text{rot}}}{8\pi} d^3\mathbf{k} = \int \frac{E_{\text{irr}}^2 + E_{\text{rot}}^2}{8\pi} d^3\mathbf{r} \equiv U_{\text{E,irr}} + U_{\text{E,rot}}. \end{aligned} \quad (28)$$

Since  $\nabla \cdot \mathbf{E}_{\text{irr}} = 4\pi\rho$  and  $\mathbf{E}_{\text{irr}} = -\nabla V^{(\text{C})}$  (Appendix B), the field energy  $U_{\text{E,irr}}$  can be transformed in the usual way to the instantaneous Coulomb energy,

$$U_{\text{E,irr}} = \int \frac{-\mathbf{E}_{\text{irr}} \cdot \nabla V^{(\text{C})}}{8\pi} d\text{Vol} = \int \frac{\rho V^{(\text{C})}}{2} d\text{Vol} = \frac{1}{2} \sum_{i \neq j} e_i V^{(\text{C})}(\mathbf{r}_i) = \frac{1}{2} \sum_{i \neq j} \frac{e_i e_j}{R_{ij}} = U^{(\text{C})}, \quad (29)$$

where  $V^{(\text{C})}(\mathbf{r}_i)$  is the instantaneous Coulomb potential at charge  $i$  due to other charges. Also, since  $\mathbf{B} = \mathbf{B}_{\text{rot}}$ , the field energy can be written as,

$$U_{\text{EM}} = U^{(\text{C})} + U_{\text{E,rot}} + U_{\text{B,rot}} = U^{(\text{C})} + U_{\text{EM,rot}}, \quad (30)$$

where  $U_{\text{EM,rot}} = U_{\text{E,rot}} + U_{\text{B,rot}}$ .

### 2.2.1 Total Energy in the Darwin Approximation

In the Darwin approximation the total energy of a system of particles of rest masses  $m_i$  and electric charges  $e_i$  is given by eq. (100),

$$U = \sum_i \frac{m_i v_i^2}{2} + \sum_i \frac{3m_i v_i^4}{8c^2} + \frac{1}{2} \sum_{i \neq j} \frac{e_i e_j}{R_{ij}} + \frac{1}{2} \sum_{i \neq j} \frac{e_i e_j}{2c^2 R_{ij}} [\mathbf{v}_i \cdot \mathbf{v}_j + (\mathbf{v}_i \cdot \hat{\mathbf{n}}_{ij})(\mathbf{v}_j \cdot \hat{\mathbf{n}}_{ij})]. \quad (31)$$

In this quasistatic approximation the rotational field energies are,

$$U_{\text{E,rot}} = 0, \quad U_{\text{M,rot}} = \frac{1}{2} \sum_i \frac{e_i \mathbf{v}_i \cdot \mathbf{A}_{\text{rot}}(\mathbf{r}_i)}{c} = \frac{1}{2} \sum_{i \neq j} \frac{e_i e_j}{2c^2 R_{ij}} [\mathbf{v}_i \cdot \mathbf{v}_j + (\mathbf{v}_i \cdot \hat{\mathbf{n}}_{ij})(\mathbf{v}_j \cdot \hat{\mathbf{n}}_{ij})], \quad (32)$$

referring to eqs. (113)-(114), where  $\mathbf{A}_{\text{rot}}(\mathbf{r}_i)$  is the rotational part of the vector potential at charge  $i$  due to other charges.

## 2.3 Total Momentum of an Electromagnetic System

The total momentum associated with electromagnetic fields  $\mathbf{E}$  and  $\mathbf{B}$  is,

$$\mathbf{P}_{\text{EM}} = \int \frac{\mathbf{S}}{c^2} d\text{Vol} = \int \frac{\mathbf{E} \times \mathbf{B}}{4\pi c} d\text{Vol}, \quad (33)$$

where  $\mathbf{S} = (c/4\pi)\mathbf{E} \times \mathbf{B}$  is the Poynting vector. We do not consider the Helmholtz decomposition of the Poynting vector, but rather a form based on the Helmholtz decomposition of the electric field,

$$\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2 = \frac{c}{4\pi}\mathbf{E}_{\text{irr}} \times \mathbf{B}_{\text{rot}} + \frac{c}{4\pi}\mathbf{E}_{\text{rot}} \times \mathbf{B}_{\text{rot}}. \quad (34)$$

Then, using the Parseval-Plancherel identity (84) and eqs. (76), (79) and (81),

$$\begin{aligned} \mathbf{P}_{\text{EM},1} &= \int \frac{\mathbf{S}_1}{c^2} d\text{Vol} = \int \frac{\mathbf{E}_{\text{irr}} \times \mathbf{B}_{\text{rot}}}{4\pi c} d^3\mathbf{r} = \int \frac{\tilde{\mathbf{E}}_{\text{irr}}^* \times \tilde{\mathbf{B}}_{\text{rot}}}{4\pi c} d^3\mathbf{k} \\ &= \int \tilde{\rho}(\mathbf{k}) \frac{4\pi i \hat{\mathbf{k}}}{k} \times \frac{i \mathbf{k} \times \tilde{\mathbf{A}}}{4\pi c} d^3\mathbf{k} = \int \frac{\tilde{\rho}(\mathbf{k}) [\tilde{\mathbf{A}} - (\tilde{\mathbf{A}} \cdot \hat{\mathbf{k}}) \hat{\mathbf{k}}]}{c} d^3\mathbf{k} \\ &= \int \frac{\tilde{\rho}(\mathbf{k}) \tilde{\mathbf{A}}_{\text{rot}}}{c} d^3\mathbf{k} = \int \frac{\rho \mathbf{A}_{\text{rot}}}{c} d^3\mathbf{r} = \sum_i \frac{e_i \mathbf{A}_{\text{rot}}(\mathbf{r}_i)}{c} = \mathbf{P}_{\text{EM,canonical}}, \end{aligned} \quad (35)$$

where  $\mathbf{A}_{\text{rot}}(\mathbf{r}_i)$  is the rotational part of vector potential at particle  $i$  due to all other charges<sup>9,10</sup>. Thus, we recognize  $\mathbf{P}_{\text{EM},1}$  as the electromagnetic part,  $\mathbf{P}_{\text{EM,canonical}}$ , of the total (gauge-invariant) canonical momentum of the system,

$$\mathbf{P}_{\text{canonical}} = \mathbf{P}_{\text{mech}} + \mathbf{P}_{\text{EM,canonical}} = \sum_i \left( \mathbf{p}_i + \frac{e_i \mathbf{A}_{\text{rot}}(\mathbf{r}_i)}{c} \right), \quad (36)$$

where  $\mathbf{p}_i = \gamma_i m_i \mathbf{v}_i$  is the (relativistic) mechanical momentum of particle  $i$ . It is often convenient to consider that the electromagnetic part of the canonical momentum of a charge is associated with the charge, although it is more correct to consider this term to be an effect of the interaction between the electromagnetic fields of that charge and the fields of other charges.

The part of the electromagnetic momentum associated with the rotational part of the electric field is,

$$\begin{aligned} \mathbf{P}_{\text{EM},2} &= \int \frac{\mathbf{E}_{\text{rot}} \times \mathbf{B}}{4\pi c} d\text{Vol} = \int \frac{\mathbf{E}_{\text{rot}} \times (\nabla \times \mathbf{A}_{\text{rot}})}{4\pi c} d\text{Vol} \\ &= \int \frac{\sum_{j=1}^3 E_{\text{rot},j} \nabla A_{\text{rot},j} - (\mathbf{E}_{\text{rot}} \cdot \nabla) \mathbf{A}_{\text{rot}}}{4\pi c} d\text{Vol} \end{aligned}$$

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<sup>9</sup>The rotational vector potential  $\mathbf{A}_{\text{rot}}$  is not unique as any gauge transformation with a gauge function  $\chi$  that obeys  $\nabla^2 \chi = 0$  leads to another rotational vector potential.

<sup>10</sup>While the vector potential  $\mathbf{A}_{\text{rot}}$  can be nonzero in situations where the electric charge density is everywhere zero, for  $\mathbf{P}_{\text{EM},1} = \mathbf{P}_{\text{EM,canonical}}$  to be nonzero requires a nonzero charge density, *i.e.*, at least one charge  $e$  not balanced by neighboring charges. Then, the electric field is nonzero, the Poynting vector is nonzero, and  $\mathbf{P}_{\text{EM}}$  according to eq. (33) is nonzero.

$$\begin{aligned}
&= \int \frac{\sum_{j=1}^3 E_{\text{rot},j} \nabla A_{\text{rot},j} + (\nabla \cdot \mathbf{E}_{\text{rot}}) \mathbf{A}_{\text{rot}}}{4\pi c} d\text{Vol} \\
&= \int \sum_{j=1}^3 \frac{E_{\text{rot},j} \nabla A_{\text{rot},j}}{4\pi c} d\text{Vol} \equiv \int \mathbf{p}_{\text{EM,orbital}} d\text{Vol} = \mathbf{P}_{\text{EM,orbital}}, \quad (37)
\end{aligned}$$

where looking ahead to eq (58) we define,

$$\mathbf{p}_{\text{EM,orbital}} = \sum_{j=1}^3 \frac{E_{\text{rot},j} \nabla A_{\text{rot},j}}{4\pi c}. \quad (38)$$

The total momentum of the system can now be written as,

$$\mathbf{P}_{\text{total}} = \mathbf{P}_{\text{canonical}} + \mathbf{P}_{\text{EM,orbital}} = \sum_i \left( \mathbf{p}_i + \frac{e_i \mathbf{A}_{\text{rot}}(\mathbf{r}_i)}{c} \right) + \int \sum_{j=1}^3 \frac{E_{\text{rot},j} \nabla A_{\text{rot},j}}{4\pi c} d\text{Vol} \quad (39)$$

It is often convenient to consider that the electromagnetic part, eq. (35), of the canonical momentum of a charge is associated with the charge, although it is more correct to consider this term to be an effect (“dressing”) of the interaction between the electromagnetic fields of that charge and the fields of other charges. In the former view, the momentum  $\mathbf{P}_{\text{EM},2}$  associated with the rotational part of the electric field is the only momentum that is “purely” associated with the fields themselves. A pulse of electromagnetic radiation that no longer overlaps with its source charges and currents can be considered as having a purely rotational electric field, such that  $\mathbf{P}_{\text{EM},2}$  describes all of the momentum of the pulse.

### 2.3.1 Momentum of a Circularly Polarized Plane Wave

As an example, consider a circularly polarized electromagnetic plane wave defined by the potentials,

$$\mathbf{A}_{\text{rot}} = A_0(\hat{\mathbf{x}} \pm i\hat{\mathbf{y}})e^{i(kz-\omega t)}, \quad V^{(C)} = 0, \quad (40)$$

for which the electromagnetic fields are,

$$\mathbf{E} = \mathbf{E}_{\text{rot}} = -\frac{1}{c} \frac{\partial \mathbf{A}_{\text{rot}}}{\partial t} = ik\mathbf{A}_{\text{rot}} = ikA_0(\hat{\mathbf{x}} \pm i\hat{\mathbf{y}})e^{i(kz-\omega t)}, \quad (41)$$

and,

$$\mathbf{B} = \nabla \times \mathbf{A}_{\text{rot}} = i\mathbf{k} \times \mathbf{A}_{\text{rot}} = \hat{\mathbf{k}} \times \mathbf{E}, \quad (42)$$

where  $\mathbf{k} = k\hat{\mathbf{z}}$ . The time-average density of electromagnetic momentum associated with the (rotational) electric field is,

$$\langle \mathbf{p}_{\text{EM},1} \rangle = 0, \quad \langle \mathbf{p}_{\text{EM},2} \rangle = \frac{1}{2} \sum_{j=1}^3 \frac{\text{Re}(E_{\text{rot},j}^* \nabla A_{\text{rot},j})}{4\pi c} = \frac{k^2 A_0^2}{4\pi c} \hat{\mathbf{k}}. \quad (43)$$

The time-average density of electromagnetic energy is,

$$\langle u \rangle = \frac{1}{2} \frac{|E|^2 + |B|^2}{8\pi} = \frac{k^2 A_0^2}{4\pi}, \quad (44)$$

so that,

$$\langle \mathbf{p}_{\text{EM}} \rangle = \frac{c}{8\pi} \text{Re}(\mathbf{E}^* \times \mathbf{B}) = \langle \mathbf{p}_{\text{EM},2} \rangle = \frac{\langle u \rangle}{c} \hat{\mathbf{k}}, \quad (45)$$

as expected.

### 2.3.2 Is There Such a Thing as “Spin Linear Momentum”?

Equations (33), (35) and (37) suggest that densities of momentum stored in an electromagnetic field can be defined as,

$$\mathbf{p}_{\text{EM}} = \mathbf{p}_{\text{EM},\text{canonical}} + \mathbf{p}_{\text{EM},\text{orbital}} = \frac{\rho \mathbf{A}_{\text{rot}}}{c} + \sum_{j=1}^3 \frac{E_{\text{rot},j} \nabla A_{\text{rot},j}}{4\pi c}, \quad (46)$$

although only the volume integrals of these densities have clear physical significance.

On comparing eqs. (38) and (57), it is suggestive to identify a density of “spin linear momentum” as,

$$\mathbf{p}_{\text{EM},\text{spin}} = -\frac{(\mathbf{E}_{\text{rot}} \cdot \nabla) \mathbf{A}_{\text{rot}}}{4\pi c}. \quad (47)$$

However, the significance of this identification is questionable, since the volume integral of  $\mathbf{p}_{\text{EM},\text{spin}}$  is zero. Furthermore,  $\mathbf{p}_{\text{EM},\text{spin}} = 0$  for a circularly polarized plane wave (40)-(42) whose characterization as carrying spin angular momentum is a primary motivation for the entire present analysis. Hence, we will not consider the notion of “spin linear momentum” further, although this concept has its advocates [18].

### 2.3.3 Total Momentum in the Darwin Approximation

In the Darwin approximation the total momentum of a system of charges is given by eq. (98),

$$\mathbf{P}_{\text{total}} = \mathbf{P}_{\text{canonical}} = \mathbf{P}_{\text{mech}} + \mathbf{P}_{\text{EM}} = \sum_i m_i \mathbf{v}_i + \sum_i \frac{m_i v_i^2}{2c^2} \mathbf{v}_i + \sum_i \frac{e_i \mathbf{A}_{\text{rot}}(\mathbf{r}_i)}{c} \quad (48)$$

In this quasistatic approximation,  $\mathbf{P}_{\text{EM},2} = 0$ , and all the electromagnetic momentum of the system can be associated with charges via the electromagnetic part of their canonical momenta, which are of order  $1/c^2$  since the vector potential is of order  $1/c$ . Only when electrodynamic effects are considered at higher orders do they include a nonzero contribution to the electromagnetic momentum from the rotational part of the electric field. For example, the (rotational) radiation fields of an oscillating dipole are of order  $1/c^2$ , so the electromagnetic momentum associated with a pulse of radiation is of order  $1/c^5$ .

Another result in the Darwin (quasistatic) approximation is based on the simplification of the wave equation (90) for the vector potential to the static equation,

$$\nabla^2 \mathbf{A}_{\text{rot}} \approx -\frac{4\pi}{c} \mathbf{J}_{\text{rot}}. \quad (49)$$



Then [19, 20],

$$\begin{aligned}
\mathbf{P}_{\text{EM,canonical}} &= \int \frac{\mathbf{E}_{\text{irr}} \times \mathbf{B}_{\text{rot}}}{4\pi c} d\text{Vol} = - \int \frac{\nabla V^{(C)} \times \mathbf{B}_{\text{rot}}}{4\pi c} d\text{Vol} = \int \frac{V^{(C)} \nabla \times \mathbf{B}_{\text{rot}}}{4\pi c} d\text{Vol} \\
&= \int \frac{V^{(C)} \nabla \times (\nabla \times \mathbf{A}_{\text{rot}})}{4\pi c} d\text{Vol} = \int \frac{V^{(C)} [\nabla(\nabla \cdot \mathbf{A}_{\text{rot}}) - \nabla^2 \mathbf{A}_{\text{rot}}]}{4\pi c} d\text{Vol} \\
&\approx \int \frac{V^{(C)} \mathbf{J}_{\text{rot}}}{c^2} d\text{Vol}, \tag{50}
\end{aligned}$$

where  $V^{(C)}$  is the instantaneous (Coulomb) potential.

### 2.3.4 Potential Momentum and “Hidden” Momentum

It is sometimes considered paradoxical that a static electromagnetic system can have nonzero electromagnetic momentum (33). See, for example, [21].

The present analysis offers the perspective that in static configurations the electric field is purely irrotational, so the electromagnetic momentum (33) can be rewritten as,

$$\mathbf{P}_{\text{EM}} = \mathbf{P}_{\text{EM,canonical}} = \sum_i \frac{e_i \mathbf{A}_{\text{rot}}(\mathbf{r}_i)}{c}. \tag{51}$$

This momentum is a kind of electrical potential momentum [22, 23] associated with a charge being at a location with nonzero vector potential (due to other sources). The potential momentum  $e\mathbf{A}/c$  of a charge  $e$  can be combined with the electrical potential energy  $eV$  of that charge, where  $V$  is the scalar potential at the location of the charge (due to other sources), into a potential energy-momentum 4-vector,

$$U_{\text{potential},\mu} = \left( eV, \frac{e\mathbf{A}}{c} \right) = (eV, e\mathbf{A}) = eA_\mu. \tag{52}$$

The implication is that if the vector potential drops to zero, the charge takes on a mechanical momentum (in addition to any initial mechanical momentum) equal to its initial electrical potential momentum.

However, this effect is obscured in many apparently simple examples because of the fact [24] that if the center of energy,

$$\mathbf{r}_U = \frac{\int \mathbf{r} u_{\text{total}} d\text{Vol}}{\int u_{\text{total}} d\text{Vol}}, \tag{53}$$

of a system with total-energy density  $u_{\text{total}}$  is at rest, then the total momentum of the system must be zero. If a static system is at rest (except for the steady currents that generate the vector potential), its center of energy will also be at rest, and the total momentum of the system must be zero. Such a system must possess a nonzero mechanical momentum equal and opposite to the electrical potential momentum (51). If the vector potential drops to zero in such a way that the center of energy remains at rest, then the mechanical momentum of the system drops to zero as well. In such cases the electrical potential momentum and the mechanical momentum are “hidden” [25].

## 2.4 Total Angular Momentum of an Electromagnetic System

The angular momentum of the electromagnetic fields of a system of charges can be written in terms of the Poynting vector as,

$$\mathbf{L}_{\text{EM}} = \int \mathbf{r} \times \frac{\mathbf{S}}{c^2} d\text{Vol} = \int \frac{\mathbf{r} \times (\mathbf{E} \times \mathbf{B})}{4\pi c} d\text{Vol}, \quad (54)$$

As for the linear momentum of the fields, it is of interest to consider separately the contribution associated with the irrotational and rotational parts of the electric field.

The part of the electromagnetic angular momentum associated with  $\mathbf{E}_{\text{irr}} = -\nabla V^{(\text{C})}$ , for which  $\nabla \cdot \mathbf{E}_{\text{rot}} = 4\pi\rho$ , is,

$$\begin{aligned} \mathbf{L}_{\text{EM},1} &= \int \frac{\mathbf{r} \times (\mathbf{E}_{\text{irr}} \times \mathbf{B})}{4\pi c} d\text{Vol} = \int \frac{\mathbf{r} \times [\mathbf{E}_{\text{irr}} \times (\nabla \times \mathbf{A}_{\text{rot}})]}{4\pi c} d\text{Vol} \\ &= \int \frac{\sum_{j=1}^3 E_{\text{irr},j} (\mathbf{r} \times \nabla) A_{\text{rot},j} - \mathbf{r} \times (\mathbf{E}_{\text{irr}} \cdot \nabla) \mathbf{A}_{\text{rot}}}{4\pi c} d\text{Vol} \\ &= \int \frac{\sum_{j=1}^3 E_{\text{irr},j} (\mathbf{r} \times \nabla) A_{\text{rot},j} - (\mathbf{E}_{\text{irr}} \cdot \nabla) (\mathbf{r} \times \mathbf{A}_{\text{rot}}) + \mathbf{E}_{\text{irr}} \times \mathbf{A}_{\text{rot}}}{4\pi c} d\text{Vol} \\ &= \int \frac{\sum_{j=1}^3 E_{\text{irr},j} (\mathbf{r} \times \nabla) A_{\text{rot},j} - (\mathbf{E}_{\text{irr}} \cdot \nabla) (\mathbf{r} \times \mathbf{A}_{\text{rot}}) + \mathbf{E}_{\text{irr}} \times \mathbf{A}_{\text{rot}}}{4\pi c} d\text{Vol} \\ &= \int \frac{\sum_{j=1}^3 E_{\text{irr},j} (\mathbf{r} \times \nabla) A_{\text{rot},j} + (\nabla \cdot \mathbf{E}_{\text{irr}}) (\mathbf{r} \times \mathbf{A}_{\text{rot}}) + \mathbf{E}_{\text{irr}} \times \mathbf{A}_{\text{rot}}}{4\pi c} d\text{Vol} \\ &= \int \frac{-\sum_{j=1}^3 (\nabla_j V^{(\text{C})}) (\mathbf{r} \times \nabla) A_{\text{rot},j} + 4\pi\rho (\mathbf{r} \times \mathbf{A}_{\text{rot}}) - (\nabla V^{(\text{C})}) \times \mathbf{A}_{\text{rot}}}{4\pi c} d\text{Vol} \\ &= \int \frac{\sum_{j=1}^3 V^{(\text{C})} \nabla_j (\mathbf{r} \times \nabla) A_{\text{rot},j} + 4\pi\rho (\mathbf{r} \times \mathbf{A}_{\text{rot}}) + V^{(\text{C})} (\nabla \times \mathbf{A}_{\text{rot}})}{4\pi c} d\text{Vol} \\ &= \int \frac{V^{(\text{C})} (\mathbf{r} \times \nabla) (\nabla \cdot \mathbf{A}_{\text{rot}}) - V^{(\text{C})} (\nabla \times \mathbf{A}_{\text{rot}}) + 4\pi\rho (\mathbf{r} \times \mathbf{A}_{\text{rot}}) + V^{(\text{C})} (\nabla \times \mathbf{A}_{\text{rot}})}{4\pi c} d\text{Vol} \\ &= \int \frac{\mathbf{r} \times \rho \mathbf{A}_{\text{rot}}}{c} d\text{Vol} = \sum_i \mathbf{r}_i \times \frac{e_i \mathbf{A}_{\text{rot}}(\mathbf{r}_i)}{c} = \sum_i \mathbf{r}_i \times \mathbf{P}_{\text{EM,canonical},i} \equiv \mathbf{L}_{\text{EM,canonical}}. \quad (55) \end{aligned}$$

assuming the various surface integrals that result from integrations by parts vanish for fields that fall off sufficiently quickly at infinity. The sum of  $\mathbf{L}_{\text{EM},1} = \mathbf{L}_{\text{EM,canonical}}$  and the mechanical angular momentum of the system is,

$$\mathbf{L}_{\text{mech}} + \mathbf{L}_{\text{EM,canonical}} = \sum_i \mathbf{r}_i \times \left( \mathbf{p}_i + \frac{e_i \mathbf{A}_{\text{rot}}(\mathbf{r}_i)}{c} \right) = \mathbf{L}_{\text{canonical}}, \quad (56)$$

which is the canonical angular momentum of the particles of the system.

Turning to the electromagnetic angular momentum associated with the rotational part of the electric field, for which  $\nabla \cdot \mathbf{E}_{\text{rot}} = 0$ , we have,

$$\mathbf{L}_{\text{EM},2} = \int \frac{\mathbf{r} \times (\mathbf{E}_{\text{rot}} \times \mathbf{B})}{4\pi c} d\text{Vol} = \int \frac{\mathbf{r} \times [\mathbf{E}_{\text{rot}} \times (\nabla \times \mathbf{A}_{\text{rot}})]}{4\pi c} d\text{Vol}$$

$$\begin{aligned}
&= \int \frac{\sum_{j=1}^3 E_{\text{rot},j}(\mathbf{r} \times \nabla) A_{\text{rot},j} - \mathbf{r} \times (\mathbf{E}_{\text{rot}} \cdot \nabla) \mathbf{A}_{\text{rot}}}{4\pi c} d\text{Vol} \\
&= \int \frac{\sum_{j=1}^3 E_{\text{rot},j}(\mathbf{r} \times \nabla) A_{\text{rot},j} - (\mathbf{E}_{\text{rot}} \cdot \nabla)(\mathbf{r} \times \mathbf{A}_{\text{rot}}) + \mathbf{E}_{\text{rot}} \times \mathbf{A}_{\text{rot}}}{4\pi c} d\text{Vol} \\
&= \int \frac{\sum_{j=1}^3 E_{\text{rot},j}(\mathbf{r} \times \nabla) A_{\text{rot},j} - (\mathbf{E}_{\text{rot}} \cdot \nabla)(\mathbf{r} \times \mathbf{A}_{\text{rot}}) + \mathbf{E}_{\text{rot}} \times \mathbf{A}_{\text{rot}}}{4\pi c} d\text{Vol} \\
&= \int \frac{\sum_{j=1}^3 E_{\text{rot},j}(\mathbf{r} \times \nabla) A_{\text{rot},j} + (\nabla \cdot \mathbf{E}_{\text{rot}})(\mathbf{r} \times \mathbf{A}_{\text{rot}}) + \mathbf{E}_{\text{rot}} \times \mathbf{A}_{\text{rot}}}{4\pi c} d\text{Vol} \\
&= \int \mathbf{r} \times \frac{\sum_{j=1}^3 E_{\text{rot},j} \nabla A_{\text{rot},j}}{4\pi c} d\text{Vol} + \int \frac{\mathbf{E}_{\text{rot}} \times \mathbf{A}_{\text{rot}}}{4\pi c} d\text{Vol} \\
&\equiv \mathbf{L}_{\text{EM,orbital}} + \mathbf{L}_{\text{EM,spin}}, \tag{57}
\end{aligned}$$

where the orbital angular momentum,

$$\mathbf{L}_{\text{EM,orbital}} = \int \mathbf{l}_{\text{EM,orbital}} d\text{Vol}, \quad \mathbf{l}_{\text{EM,orbital}} = \mathbf{r} \times \frac{\sum_{j=1}^3 E_{\text{rot},j} \nabla A_{\text{rot},j}}{4\pi c} = \mathbf{r} \times \mathbf{p}_{\text{EM,orbital}}, \tag{58}$$

depends on the choice of origin, while,

$$\mathbf{L}_{\text{EM,spin}} = \int \mathbf{l}_{\text{EM,spin}} d\text{Vol}, \quad \mathbf{l}_{\text{EM,spin}} = \frac{\mathbf{E}_{\text{rot}} \times \mathbf{A}_{\text{rot}}}{4\pi c} \tag{59}$$

is independent of the choice of origin and is therefore an intrinsic property of the fields, which we call the spin angular momentum.

### 2.4.1 Angular Momentum of a Circularly Polarized Plane Wave

As an example, consider a circularly polarized electromagnetic plane wave, eqs. (40)-(42). The time-average density of spin angular momentum is,

$$\langle \mathbf{l}_{\text{EM,spin}} \rangle = \frac{1}{2} \frac{\text{Re}(\mathbf{E}_{\text{rot}}^* \times \mathbf{A}_{\text{rot}})}{4\pi c} = \pm \frac{k A_0^2}{4\pi c} \hat{\mathbf{k}}. \tag{60}$$

Thus,

$$\langle \mathbf{l}_{\text{EM,spin}} \rangle = \pm \frac{\langle u \rangle}{\omega} \hat{\mathbf{k}}, \tag{61}$$

in terms of the time-average density (44) of electromagnetic energy, which is consistent with the quantum behavior of spin-1 photons. Also, the time-average density of orbital angular momentum is,

$$\langle \mathbf{l}_{\text{EM,orbital}} \rangle = \mathbf{r} \times \frac{1}{2} \sum_j \frac{\text{Re}(E_{\text{rot},j}^* \nabla A_{\text{rot},j})}{4\pi c} = \mathbf{r} \times \langle \mathbf{p}_{\text{EM}} \rangle. \tag{62}$$

recalling eq. (45).

This illustrates that for any free field, for which the ‘‘orbital’’ momentum density equals the total momentum density, we might expect that the ‘‘orbital’’ angular momentum density is the total angular momentum density, which brings into question the significance of the ‘‘spin’’ angular momentum density.

### 2.4.2 Is There “Really” Such a Thing as Classical Spin Angular Momentum?

Equation (54) suggests that we could define the density of angular momentum in the electromagnetic field as,

$$\mathbf{l}_{\text{EM}} = \mathbf{r} \times \frac{\mathbf{E} \times \mathbf{B}}{4\pi c}. \quad (63)$$

Then, eq. (46) suggests that we can replace  $\mathbf{E} \times \mathbf{B}/4\pi c$  by  $\mathbf{p}_{\text{EM,canonical}} + \mathbf{p}_{\text{EM,orbital}}$  to write,

$$\mathbf{l}_{\text{EM}} \stackrel{?}{=} \mathbf{r} \times (\mathbf{p}_{\text{EM,canonical}} + \mathbf{p}_{\text{EM,orbital}}). \quad (64)$$

In contrast, eqs. (55)-(59) suggest that we can also write,

$$\begin{aligned} \mathbf{l}_{\text{EM}} &= \mathbf{l}_{\text{EM,canonical}} + \mathbf{l}_{\text{EM,orbital}} + \mathbf{l}_{\text{EM,spin}} = \mathbf{r} \times (\mathbf{p}_{\text{EM,canonical}} + \mathbf{p}_{\text{EM,orbital}}) + \mathbf{l}_{\text{EM,spin}} \\ &= \mathbf{r} \times \frac{\rho \mathbf{A}_{\text{rot}}}{c} + \mathbf{r} \times \frac{\sum_{j=1}^3 E_{\text{rot},j} \nabla A_{\text{rot},j}}{4\pi c} + \frac{\mathbf{E}_{\text{rot}} \times \mathbf{A}_{\text{rot}}}{4\pi c}. \end{aligned} \quad (65)$$

The analysis that has led to the apparent contradiction between eqs. (64) and (65) assumed that the surface integrals that arise during the various integrations by parts can be neglected. This assumption is not valid for plane waves, or for monochromatic waves whose time dependence  $e^{-i\omega t}$  implies these waves exist at arbitrarily early and late times. Physical waves have existed only for a finite time, and hence are bounded in space such that the surface integrals are indeed negligible. That is, neglect of the integrals on distant surfaces is a good approximation for physics fields.

Thus, the transformations (35), (37), (55) and (57) do not justify equating the integrand  $\mathbf{E} \times \mathbf{B}/4\pi c$  with  $\mathbf{p}_{\text{EM,canonical}} + \mathbf{p}_{\text{EM,orbital}}$ , or equating the integrand  $\mathbf{r} \times (\mathbf{E} \times \mathbf{B})/4\pi c$  to the form  $\mathbf{r} \times (\mathbf{p}_{\text{EM,canonical}} + \mathbf{p}_{\text{EM,orbital}}) + \mathbf{l}_{\text{EM,spin}}$ . In particular, the argument that led to eq. (64) does not imply that the volume integral of  $\mathbf{r} \times (\mathbf{p}_{\text{EM,canonical}} + \mathbf{p}_{\text{EM,orbital}})$  equals the total electromagnetic angular momentum (54) of the system. While care must be taken when using the densities of momentum and angular momentum introduced here (and elsewhere), there remains a valid domain of applicability of these concepts, including the “spin” angular momentum density (59).

A related issue is how we should regard the two forms of angular momentum density (63) and (65), both of whose volume integrals yield that same total electromagnetic angular momentum for a bounded system. The form (63) suggests that all electromagnetic angular momentum is “orbital”, while the form (65) includes the “intrinsic spin” angular momentum (54).

*The situation here is similar to that concerning magnetostatics, where a classical model of, say, iron atoms is that each has a magnetic moment related to the microscopic current density  $\mathbf{J}_{\text{atom}}$  within the atom,*

$$\begin{aligned} \mathbf{M}_{\text{atom}} &= \frac{1}{2c} \int_{\text{atom}} (\mathbf{r} - \mathbf{r}_{\text{atom}}) \times \mathbf{J}_{\text{atom}} d\text{Vol} = \int_{\text{atom}} \frac{\mathbf{r} \times \mathbf{J}_{\text{atom}}}{2c} d\text{Vol} + \frac{\mathbf{r}_{\text{atom}}}{2c} \times \int \mathbf{J}_{\text{atom}} d\text{Vol} \\ &= \int_{\text{atom}} \frac{\mathbf{r} \times \mathbf{J}_{\text{atom}}}{2c} d\text{Vol}, \end{aligned} \quad (66)$$

which is independent of the choice of origin for steady current distributions  $\mathbf{J}_{\text{atom}}$ .<sup>11</sup> A ferromagnetic magnetic moment is considered to be an intrinsic property of the atom (and related to the “spin” angular momentum of the atom). We can calculate the total magnetic moment of a block of iron as the sum of all atomic moments, which can be transformed into an integral over the macroscopic current density  $\mathbf{J}$ ,

$$\mathbf{M}_{\text{total}} = \sum_{\text{atoms}} \mathbf{M}_{\text{atom}} = \frac{1}{2c} \int \mathbf{r} \times \sum_{\text{atoms}} \mathbf{J}_{\text{atom}} d\text{Vol} = \int \frac{\mathbf{r} \times \mathbf{J}}{2c} d\text{Vol}, \quad (67)$$

where  $\mathbf{J}$  is obtained by averaging the atoms currents  $\mathbf{J}_{\text{atom}}$  over volumes large compared to an atom but small compared to macroscopic scales. We now can define magnetization densities in two ways, microscopic and macroscopic:

$$\mathbf{m}_{\text{micro}} = \frac{\mathbf{M}_{\text{atom}}}{\text{Vol}_{\text{atom}}}, \quad \text{and} \quad \mathbf{m}_{\text{macro}} = \frac{\mathbf{r} \times \mathbf{J}}{2c}, \quad (68)$$

such that,

$$\mathbf{M}_{\text{total}} = \int \mathbf{m}_{\text{micro}} d\text{Vol} = \int \mathbf{m}_{\text{macro}} d\text{Vol}. \quad (69)$$

However, the microscopic and macroscopic magnetization densities are very different; a uniform microscopic density is associated with a macroscopic density that is nonzero only on the surface of the iron block.

Returning to the case of electromagnetic angular momentum, we can certainly consider the form (63) to represent the macroscopic density of electromagnetic angular momentum.<sup>12</sup> It is appealing to argue that the form (65) corresponds to a more microscopic description, in which the intrinsic angular momentum of “particles” of the electromagnetic field is described by the density (59) of “spin” angular momentum. Such an interpretation is not entirely justified by the usual premises of classical electrodynamics, but it is more acceptable from a quantum perspective.<sup>13,14</sup>

## Appendix A: Fourier Transforms

The Fourier transform of a vector field  $\mathbf{F}(\mathbf{r})$  in ordinary 3-space is the vector field  $\tilde{\mathbf{F}}(\mathbf{k})$  in  $k$ -space defined by,

$$\tilde{\mathbf{F}}(\mathbf{k}) = \frac{1}{(2\pi)^{3/2}} \int \mathbf{F}(\mathbf{r}) e^{-i\mathbf{k}\cdot\mathbf{r}} d^3\mathbf{r}, \quad (70)$$

---

<sup>11</sup>Noting that  $\nabla \cdot (x_i \mathbf{J}) = \mathbf{J} \cdot \nabla x_i = J_i$ , we have that  $\int J_i d\text{Vol} = \int \nabla \cdot (x_i \mathbf{J}) d\text{Vol} = \oint (x_i \mathbf{J}) \cdot d\mathbf{Area} = 0$  for any current distribution that is bounded in space.

<sup>12</sup>Comparison with the case of a uniformly magnetized block of iron suggests that the macroscopic angular momentum of an electromagnetic field with uniform “spin” angular momentum resides on the surface of the field. In the case of a circularly polarized plane wave, the “surface” is at infinity, such that the macroscopic description omits the angular momentum by the neglect of the surface integrals.

<sup>13</sup>It is noteworthy that the formalism of “spin” electromagnetic field angular momentum arose in the context of classical field theories [6, 8] of particles with “spin”.

<sup>14</sup>For recent comments on this theme, see [26].

and the corresponding Fourier integral representation of  $\mathbf{F}$  is,

$$\mathbf{F}(\mathbf{r}) = \frac{1}{(2\pi)^{3/2}} \int \tilde{\mathbf{F}}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}} d^3\mathbf{k}. \quad (71)$$

We symbolize the relations (70)-(71) by,

$$\mathbf{F}(\mathbf{r}) \leftrightarrow \mathcal{F}(\mathbf{k}). \quad (72)$$

For example,

$$\frac{1}{r} \leftrightarrow \frac{1}{(2\pi)^{3/2}} \frac{4\pi}{k^2}, \quad \text{and} \quad \frac{\hat{\mathbf{r}}}{r^2} \leftrightarrow \frac{1}{(2\pi)^{3/2}} \frac{-4\pi i \hat{\mathbf{k}}}{k}, \quad (73)$$

where  $\hat{\mathbf{a}}$  is the unit vector  $\mathbf{a}/a$ .

The curl and divergence of the field  $\mathbf{F}$  have Fourier transforms,

$$\nabla \odot \mathbf{F} = \frac{1}{(2\pi)^{3/2}} \int \nabla \odot (\tilde{\mathbf{F}}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}}) d^3\mathbf{k} = \frac{1}{(2\pi)^{3/2}} \int i\mathbf{k} \odot \tilde{\mathbf{F}} d^3\mathbf{k}, \quad (74)$$

where  $\odot$  represents either operation  $\cdot$  or  $\times$ , which implies the relations,

$$\nabla \times \mathbf{F} \leftrightarrow i\mathbf{k} \times \tilde{\mathbf{F}}, \quad \nabla \cdot \mathbf{F} \leftrightarrow i\mathbf{k} \cdot \tilde{\mathbf{F}}, \quad (75)$$

For example,

$$\mathbf{B} = \nabla \times \mathbf{A} \leftrightarrow \tilde{\mathbf{B}} = i\mathbf{k} \times \tilde{\mathbf{A}}. \quad (76)$$

Then, from eq. (6) the Fourier transforms  $\tilde{\mathbf{F}}_{\text{irr}}$  and  $\tilde{\mathbf{F}}_{\text{rot}}$  of the irrotational and rotational parts,  $\mathbf{F}_{\text{irr}}$  and  $\mathbf{F}_{\text{rot}}$  of a vector field  $\mathbf{F}$  obey,

$$\mathbf{k} \times \tilde{\mathbf{F}}_{\text{irr}} = 0, \quad \mathbf{k} \cdot \tilde{\mathbf{F}}_{\text{rot}} = 0, \quad \tilde{\mathbf{F}}_{\text{irr}} \cdot \tilde{\mathbf{F}}_{\text{rot}} = 0, \quad (77)$$

which together with the relation,

$$\mathbf{F} = \mathbf{F}_{\text{irr}} + \mathbf{F}_{\text{rot}} \leftrightarrow \tilde{\mathbf{F}} = \tilde{\mathbf{F}}_{\text{irr}} + \tilde{\mathbf{F}}_{\text{rot}} \quad (78)$$

imply that,

$$\tilde{\mathbf{F}}_{\text{irr}} = (\tilde{\mathbf{F}} \cdot \hat{\mathbf{k}}) \hat{\mathbf{k}} = \tilde{\mathbf{F}}_{\parallel}, \quad \tilde{\mathbf{F}}_{\text{rot}} = \tilde{\mathbf{F}} - \tilde{\mathbf{F}}_{\text{irr}} = \tilde{\mathbf{F}}_{\perp}. \quad (79)$$

As an example, the Maxwell equation (14) has the Fourier transform,

$$i\mathbf{k} \cdot \tilde{\mathbf{E}} = 4\pi\tilde{\rho}(\mathbf{k}), \quad (80)$$

where  $\tilde{\rho}(\mathbf{k})$  is the transform of  $\rho(\mathbf{r})$ , so the irrotational part of  $\tilde{\mathbf{E}}$  is,

$$\tilde{\mathbf{E}}_{\text{irr}} = \tilde{\rho}(\mathbf{k}) \frac{-4\pi i \hat{\mathbf{k}}}{k}, \quad (81)$$

which is the product of two Fourier transforms,  $\tilde{F} = \tilde{\rho}(\mathbf{k})$  and  $\tilde{\mathbf{G}} = -4\pi i \hat{\mathbf{k}}/k$ . In general, the product  $\tilde{F}(\mathbf{k})\tilde{G}(\mathbf{k})$  of the Fourier transforms of scalar fields  $F(\mathbf{r})$  and  $G(\mathbf{r})$  has the inverse transform,

$$\frac{1}{(2\pi)^{3/2}} \int F(\mathbf{r}')G(\mathbf{r} - \mathbf{r}') d^3\mathbf{r}', \quad (82)$$

which is not  $F(\mathbf{r})G(\mathbf{r})$  but their spatial convolution. Using eqs. (81)-(82) together with eq. (73), we find the irrotational part of the electric field to be,

$$\mathbf{E}_{\text{irr}} = \int \frac{\rho(\mathbf{r}')\hat{\mathbf{R}}}{R^2} d^3\mathbf{r}' = \mathbf{E}^{(C)}, \quad (83)$$

where  $\mathbf{R} = \mathbf{r} - \mathbf{r}'$ . Thus, the irrotational part of the electric field  $\mathbf{E}$  at time  $t$  is the instantaneous Coulomb field  $\mathbf{E}^{(C)}$  of the electric charge density  $\rho(\mathbf{r}, t)$ , *i.e.*, its “static” part, as would hold if the present charge density had never been different in the past.

We also note the Parseval-Plancherel identity for two scalar fields  $F(\mathbf{r})$  and  $G(\mathbf{r})$  with Fourier transforms  $\tilde{F}(\mathbf{k})$  and  $\tilde{G}(\mathbf{k})$ :

$$\int F^*(\mathbf{r})G(\mathbf{r}) d^3\mathbf{r} = \int \tilde{F}^*(\mathbf{k})\tilde{G}(\mathbf{k}) d^3\mathbf{k}. \quad (84)$$

## Appendix B: Coulomb Gauge

The vector potential in the Coulomb gauge is chosen to be purely rotational/transverse,

$$\nabla \cdot \mathbf{A}^{(C)} = 0, \quad \text{so that,} \quad \mathbf{A}^{(C)} = \mathbf{A}_{\text{rot}} \quad (\text{Coulomb}). \quad (85)$$

We restrict our discussion to media for which the relative permittivity is  $\epsilon = 1$  and the relative permeability is  $\mu = 1$ . Then, using eq. (16) in the Maxwell equation  $\nabla \cdot \mathbf{E} = 4\pi$ , the scalar potential  $V$  in any gauge obeys,

$$\nabla^2 V + \frac{\partial}{\partial t} \nabla \cdot \mathbf{A} = -4\pi\rho, \quad (86)$$

and the Maxwell equation  $\nabla \times \mathbf{B} = (4\pi/c)\mathbf{J} + \partial\mathbf{E}/\partial ct$  leads to,

$$\nabla^2 \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\frac{4\pi}{c} \mathbf{J} + \nabla \left( \nabla \cdot \mathbf{A} + \frac{1}{c} \frac{\partial V}{\partial t} \right) \quad (87)$$

for the vector potential in any gauge.

Thus, in the Coulomb gauge, eq. (86) becomes Poisson’s equation,

$$\nabla^2 V^{(C)} = -4\pi\rho, \quad (88)$$

which has the formal solution,

$$V^{(C)}(\mathbf{r}, t) = \int \frac{\rho(\mathbf{r}', t)}{R} d\text{Vol}' \quad (\text{Coulomb}), \quad (89)$$

where  $R = |\mathbf{r} - \mathbf{r}'|$ , in which changes in the charge distribution  $\rho$  instantaneously affect the potential  $V^{(C)}$  at any distance.

In the Coulomb gauge, eq. (87) becomes, using the continuity equation,  $\nabla \cdot \mathbf{J} + \partial\rho/\partial t = 0$ ,

$$\begin{aligned} \nabla^2 \mathbf{A}^{(C)} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}^{(C)}}{\partial t^2} &= -\frac{4\pi}{c} \mathbf{J} + \frac{\nabla}{c} \frac{\partial V^{(C)}}{\partial t} = -\frac{4\pi}{c} \mathbf{J} - \frac{4\pi}{c} \nabla \int \frac{\nabla' \cdot \mathbf{J}(\mathbf{r}', t)}{4\pi R} d\text{Vol}' \\ &= -\frac{4\pi}{c} (\mathbf{J} - \mathbf{J}_{\text{irr}}) = -\frac{4\pi}{c} \mathbf{J}_{\text{rot}}, \end{aligned} \quad (90)$$

using eqs. (13), (89) and the continuity equation,  $\nabla \cdot \mathbf{J} = -\partial\rho/\partial t$ . Thus, a formal solution for the (retarded, rotational) vector potential in the Coulomb gauge is,<sup>15</sup>

$$\mathbf{A}_{\text{rot}}^{(\text{C})}(\mathbf{r}, t) = \mathbf{A}^{(\text{C})}(\mathbf{r}, t) = \frac{1}{c} \int \frac{\mathbf{J}_{\text{rot}}(\mathbf{r}', t' = t - R/c)}{R} d\text{Vol}' \quad (\text{Coulomb}), \quad (92)$$

where the rotational part of the current density is given by,

$$\mathbf{J}_{\text{rot}}(\mathbf{r}, t) = \frac{1}{4\pi} \nabla \times \int \frac{\nabla' \times \mathbf{J}(\mathbf{r}', t)}{R} d\text{Vol}' = \frac{1}{4\pi} \nabla \times \nabla \times \int \frac{\mathbf{J}(\mathbf{r}', t)}{R} d\text{Vol}', \quad (93)$$

which goes to zero at large distance if the current density lies within a bounded volume.

## Appendix C: Darwin's Approximation

The Lagrangian for a charge  $e$  of mass  $m$  that moves with velocity  $\mathbf{v}$  in an external electromagnetic field that is described by (Coulomb-gauge) potentials  $V^{(\text{C})}$  and  $\mathbf{A}^{(\text{C})}$  can be written (see, for example, sec. 16 of [27]),

$$\mathcal{L} = -mc^2 \sqrt{1 - v^2/c^2} - eV^{(\text{C})} + e \frac{\mathbf{v}}{c} \cdot \mathbf{A}^{(\text{C})}. \quad (94)$$

Darwin [28] worked in the Coulomb gauge, and kept term only to order  $v^2/c^2$ . Then, the scalar and vector potentials due to a charge  $e$  that has velocity  $\mathbf{v}$  can be taken as (see sec. 65 of [27] or sec. 12.6 of [29]),

$$V^{(\text{C})} = \frac{e}{R}, \quad \mathbf{A}^{(\text{C})} = \frac{e[\mathbf{v} + (\mathbf{v} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}}]}{2cR}, \quad (95)$$

where  $\hat{\mathbf{n}}$  is directed from the charge to the observer, whose (present) distance is  $R$ .

Combining equations (94) and (95) for a collections of charged particles, and keeping terms only to order  $v^2/c^2$ , we arrive at the Darwin Lagrangian,

$$\mathcal{L} = \sum_i \frac{m_i v_i^2}{2} + \sum_i \frac{m_i v_i^4}{8c^2} - \sum_{i>j} \frac{e_i e_j}{R_{ij}} + \sum_{i>j} \frac{e_i e_j}{2c^2 R_{ij}} [\mathbf{v}_i \cdot \mathbf{v}_j + (\mathbf{v}_i \cdot \hat{\mathbf{n}}_{ij})(\mathbf{v}_j \cdot \hat{\mathbf{n}}_{ij})], \quad (96)$$

where we ignore the constant sum of the rest energies of the particles.

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<sup>15</sup>In case of steady currents,  $\nabla \cdot \mathbf{J} = 0$ ,  $\mathbf{J}_{\text{rot}} = \mathbf{J}$ , and a formal solution is,

$$\mathbf{A}^{(\text{C})}(\mathbf{r}) = \frac{1}{c} \int \frac{\mathbf{J}(\mathbf{r}')}{R} d\text{Vol}' \quad (\text{Coulomb, static}). \quad (91)$$

Other forms of the Coulomb-gauge vector potential are possible, via the restricted gauge transformation  $\mathbf{A}' = \mathbf{A} + \nabla\chi$ , where the gauge-transformation function obeys Laplace's equation,  $\nabla^2\chi = 0$ . However, the uniqueness theorem for solutions to Laplace's equation with Neumann boundary conditions (see sec. 1.9 of [29]), the only possible  $\chi$  for which  $\nabla\chi = 0$  at large distances is the trivial case  $\chi = 0$ . Hence, any alternative Coulomb-gauge vector potential does not go to zero everywhere at infinity, and the static form (91) plays a somewhat special role.



The Lagrangian (96) does not depend explicitly on time, so the corresponding Hamiltonian,

$$\begin{aligned}\mathcal{H} &= \sum_i \mathbf{p}_i \cdot \mathbf{v}_i - \mathcal{L} \\ &= \sum_i \frac{p_i^2}{2m_i} - \sum_i \frac{p_i^4}{8m_i^3 c^2} + \sum_{i>j} \frac{e_i e_j}{R_{ij}} - \sum_{i>j} \frac{e_i e_j}{2m_i m_j c^2 R_{ij}} [\mathbf{p}_i \cdot \mathbf{p}_j + (\mathbf{p}_i \cdot \hat{\mathbf{n}}_{ij})(\mathbf{p}_j \cdot \hat{\mathbf{n}}_{ij})]\end{aligned}\quad (97)$$

is the conserved energy of the system, where,

$$\begin{aligned}\mathbf{p}_i &= \frac{\partial \mathcal{L}}{\partial \mathbf{v}_i} = m_i \mathbf{v}_i + \frac{m_i v_i^2}{2c^2} \mathbf{v}_i + \sum_{j \neq i} \frac{e_i e_j}{2c^2 R_{ij}} [\mathbf{v}_j + \hat{\mathbf{n}}_{ij}(\mathbf{v}_j \cdot \hat{\mathbf{n}}_{ij})] \\ &= m_i \mathbf{v}_i + \frac{m_i v_i^2}{2c^2} \mathbf{v}_i + \frac{e_i \mathbf{A}^{(C)}(\mathbf{r}_i)}{c}\end{aligned}\quad (98)$$

is the canonical momentum of particle  $i$ , and,

$$\mathbf{A}^{(C)}(\mathbf{r}_i) = \sum_{j \neq i} \frac{e_j}{2c^2 R_{ij}} [\mathbf{v}_j + \hat{\mathbf{n}}_{ij}(\mathbf{v}_j \cdot \hat{\mathbf{n}}_{ij})]\quad (99)$$

is the vector potential at charge  $i$  due to the other charges. Hence, the energy/Hamiltonian is,

$$U = \sum_i \frac{m_i v_i^2}{2} + \sum_i \frac{3m_i v_i^4}{8c^2} + \sum_{i>j} \frac{e_i e_j}{R_{ij}} + \sum_{i>j} \frac{e_i e_j}{2c^2 R_{ij}} [\mathbf{v}_i \cdot \mathbf{v}_j + (\mathbf{v}_i \cdot \hat{\mathbf{n}}_{ij})(\mathbf{v}_j \cdot \hat{\mathbf{n}}_{ij})],\quad (100)$$

as first derived by Darwin [28].

The part of this Hamiltonian/energy associated with electromagnetic interactions is,

$$\begin{aligned}U_{\text{EM}} &= \frac{1}{2} \sum_{i \neq j} \frac{e_i e_j}{R_{ij}} + \frac{1}{2} \sum_{i \neq j} \frac{e_i e_j}{2c^2 R_{ij}} [\mathbf{v}_i \cdot \mathbf{v}_j + (\mathbf{v}_i \cdot \hat{\mathbf{n}}_{ij})(\mathbf{v}_j \cdot \hat{\mathbf{n}}_{ij})] \\ &= \frac{1}{2} \sum_i e_i \left( V^{(C)}(\mathbf{r}_i) + \frac{\mathbf{v}_i \cdot \mathbf{A}^{(C)}(\mathbf{r}_i)}{c} \right),\end{aligned}\quad (101)$$

where

$$V^{(C)}(\mathbf{r}_i) = \sum_{j \neq i} \frac{e_j}{R_{ij}}\quad (102)$$

is the electric scalar potential at charge  $i$  due to other charges.<sup>16</sup>

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<sup>16</sup>The integral form of eq. (101),

$$U_{\text{EM}} = \frac{1}{2} \int \left( \rho V^{(C)} + \frac{\mathbf{J} \cdot \mathbf{A}^{(C)}}{c} \right) d\text{Vol},\quad (103)$$

shows the possibly surprising result that the electromagnetic energy in the Darwin approximation has the form of that for a system of quasistatic charge and current densities  $\rho$  and  $\mathbf{J}$  (which implies use of the Coulomb gauge; see, for example, sec. 5.16 of [29] or secs. 31 and 33 of [30]).

## C.1: Direct Calculation of the Interaction Electromagnetic Energy in the Darwin Approximation

The interaction electromagnetic energy associated with a set  $\{i\}$  of charges  $e_i$  can be written,

$$U_{\text{EM}} = \sum_{i>j} \int \frac{\mathbf{E}_i \cdot \mathbf{E}_j + \mathbf{B}_i \cdot \mathbf{B}_j}{4\pi} d\text{Vol}. \quad (104)$$

The electric and magnetic fields of a charge  $e$  at distance  $R$  from an observer follow in the Darwin approximation from the potentials (98),

$$\begin{aligned} \mathbf{E} &= -\nabla V^{(C)} - \frac{\partial \mathbf{A}^{(C)}}{\partial ct} = \frac{e}{R^2} \hat{\mathbf{n}} - \frac{e}{2c^2 R} \left[ \mathbf{a} + (\mathbf{a} \cdot \hat{\mathbf{n}}) \hat{\mathbf{n}} + \frac{3(\mathbf{v} \cdot \hat{\mathbf{n}})^2 - v^2}{R} \hat{\mathbf{n}} \right] \\ &\equiv \mathbf{E}^{(C)} + \mathbf{E}_{\text{rot}}, \end{aligned} \quad (105)$$

$$\mathbf{B} = \nabla \times \mathbf{A}^{(C)} = \frac{e\mathbf{v} \times \hat{\mathbf{n}}}{cR^2}, \quad (106)$$

where  $\mathbf{a} = d\mathbf{v}/dt$  is the (present) acceleration of the charge,<sup>17</sup> and,

$$\mathbf{E}^{(C)} = \frac{e}{R^2} \hat{\mathbf{n}}, \quad \mathbf{E}_{\text{rot}} = -\frac{e}{2c^2 R} \left[ \mathbf{a} + (\mathbf{a} \cdot \hat{\mathbf{n}}) \hat{\mathbf{n}} + \frac{3(\mathbf{v} \cdot \hat{\mathbf{n}})^2 - v^2}{R} \hat{\mathbf{n}} \right]. \quad (107)$$

See [31] for applications of these relations to considerations of electromagnetic momentum rather than energy.

The potentials (95) are in the Coulomb gauge, so that  $\nabla \cdot \mathbf{A}^{(C)} = 0$ , and hence,

$$\nabla \cdot \mathbf{E}_{\text{rot}} = 0. \quad (108)$$

The electric part of the energy (100) can be written,

$$U_{\text{E}} = \sum_{i>j} e_i e_j \int \frac{\hat{\mathbf{n}}_i \cdot \hat{\mathbf{n}}_j}{4\pi R_i^2 R_j^2} d\text{Vol} + \sum_{i>j} \int \left( \frac{e_i \hat{\mathbf{n}}_i \cdot \mathbf{E}'_j}{4\pi R_i^2} + \frac{e_j \hat{\mathbf{n}}_j \cdot \mathbf{E}'_i}{4\pi R_j^2} \right) d\text{Vol} + \mathcal{O}\left(\frac{1}{c^4}\right). \quad (109)$$

It is well known (see, for example, the Appendix of [32]), that,

$$\int \frac{\hat{\mathbf{n}}_i \cdot \hat{\mathbf{n}}_j}{4\pi R_i^2 R_j^2} d\text{Vol} = \frac{1}{R_{ij}}. \quad (110)$$

For the second integral in eq. (105), we integrate by parts to find,<sup>18</sup>

$$\int \frac{\hat{\mathbf{n}}_i \cdot \mathbf{E}_{\text{rot},j}}{R_i^2} d\text{Vol} = - \int \mathbf{E}_{\text{rot},j} \cdot \nabla \left( \frac{1}{R_i} \right) d\text{Vol} = \int \frac{1}{R_i} \nabla \cdot \mathbf{E}_{\text{rot},j} d\text{Vol} = 0. \quad (112)$$

<sup>17</sup>Sec. 65 of [27] shows that in the Darwin approximation the Liénard-Wiechert potentials (Lorenz gauge) reduce to  $V^{(L)} = e/R + (e/2c^2)\partial^2 R/\partial t^2$  and  $\mathbf{A}^{(L)} = e\mathbf{v}/cR$ , from which eqs. (101)-(103) also follow.

<sup>18</sup>The surface integral resulting from the integration by parts in eq. (112) vanishes as follows:

$$\int \frac{\mathbf{E}_{\text{rot},j}}{R_i} \cdot d\text{Area} = - \int \frac{[\mathbf{a}_j + (\mathbf{a}_j \cdot \hat{\mathbf{n}}) \hat{\mathbf{n}}]}{2c^2 R_i R_j} \cdot d\text{Area} + \int \frac{(\dots)}{R_i R_j^2} \cdot d\text{Area} \rightarrow - \int \frac{\mathbf{a}_j \cdot \hat{\mathbf{n}}}{c^2} d\Omega = 0. \quad (111)$$

Thus, the electric part of the interaction energy is,

$$U_E = \sum_{i>j} \frac{e_i e_j}{R_{ij}}, \quad (113)$$

which holds for charges of any velocity when we work in the Coulomb gauge.

The magnetic part of the energy (100) is,

$$\begin{aligned} U_M &= \sum_{i>j} \int \frac{\mathbf{B}_i \cdot \mathbf{B}_j}{4\pi} d\text{Vol} = \sum_{i>j} \int \frac{\mathbf{B}_i \cdot \nabla \times \mathbf{A}_j^{(C)}}{4\pi} d\text{Vol} = \sum_{i>j} \int \frac{\mathbf{A}_j^{(C)} \cdot \nabla \times \mathbf{B}_i}{4\pi} d\text{Vol} \\ &= \sum_{i>j} \frac{e_i \mathbf{v}_i \cdot \mathbf{A}_j^{(C)}(\mathbf{r}_i)}{c} = \sum_{i>j} \frac{e_i e_j}{2c^2 R_{ij}} [\mathbf{v}_i \cdot \mathbf{v}_j + (\mathbf{v}_i \cdot \hat{\mathbf{n}}_{ij})(\mathbf{v}_j \cdot \hat{\mathbf{n}}_{ij})], \end{aligned} \quad (114)$$

where we note that  $\mathbf{B} \cdot \nabla \times \mathbf{A} = \epsilon_{lmn} B_l \partial A_n / \partial x_m$ , so that integration by parts leads to  $-\epsilon_{lmn} A_n B_m \partial B_l / \partial x_m = \epsilon_{nml} A_n \partial B_l / \partial x_m = \mathbf{A} \cdot \nabla \times \mathbf{B}$  (and not to  $-\mathbf{A} \cdot \nabla \times \mathbf{B}$ ), and that,<sup>19</sup>

$$\nabla \times \mathbf{B}_i = \frac{4\pi}{c} \mathbf{J}_i + \frac{\partial \mathbf{E}_i}{\partial ct} = \frac{4\pi e_i \mathbf{v}_i}{c} \delta(\mathbf{r} - \mathbf{r}_i) - \nabla \frac{\partial V_i^{(C)}}{\partial ct} - \frac{\partial^2 \mathbf{A}_i^{(C)}}{\partial (ct)^2}. \quad (116)$$

Thus, we again find the interaction electromagnetic energy  $U_{EM} = U_E + U_M$  to be given by eq. (98).

## References

- [1] J.H. Poynting, *On the Transfer of Energy in the Electromagnetic Field*, Phil. Trans. Roy. Soc. London **175**, 343 (1884),  
[http://physics.princeton.edu/~mcdonald/examples/EM/poynting\\_ptrsl\\_175\\_343\\_84.pdf](http://physics.princeton.edu/~mcdonald/examples/EM/poynting_ptrsl_175_343_84.pdf)
- [2] M. Abraham, *Prinzipien der Dynamik des Elektrons*, Ann. Phys. **10**, 105 (1903),  
[http://physics.princeton.edu/~mcdonald/examples/EM/abraham\\_ap\\_10\\_105\\_03.pdf](http://physics.princeton.edu/~mcdonald/examples/EM/abraham_ap_10_105_03.pdf)
- [3] H. Helmholtz, *Über Integrale der hydrodynamischen Gleichungen, welche den Wirbelbewegungen entsprechen*, Crelles J. **55**, 25 (1858),  
[http://physics.princeton.edu/~mcdonald/examples/fluids/helmholtz\\_jram\\_55\\_25\\_58.pdf](http://physics.princeton.edu/~mcdonald/examples/fluids/helmholtz_jram_55_25_58.pdf)  
[http://physics.princeton.edu/~mcdonald/examples/fluids/helmholtz\\_jram\\_55\\_25\\_58\\_english.pdf](http://physics.princeton.edu/~mcdonald/examples/fluids/helmholtz_jram_55_25_58_english.pdf)  
*On Integrals of the Hydrodynamical Equations, which express Vortex-motion*, Phil. Mag. **33**, 485 (1867),  
[http://physics.princeton.edu/~mcdonald/examples/fluids/helmholtz\\_pm\\_33\\_485\\_67.pdf](http://physics.princeton.edu/~mcdonald/examples/fluids/helmholtz_pm_33_485_67.pdf)

<sup>19</sup>In greater detail, the integrand  $\mathbf{A}_j \cdot \nabla \times \mathbf{B}_i$  includes the term  $\mathbf{A}_j \cdot \partial^2 \mathbf{A}_i / \partial (ct)^2$  which is of order  $1/c^4$ , while the integral of the term  $\mathbf{A}_j \cdot \nabla \partial \phi_i / \partial ct$  vanishes according to,

$$-\int \mathbf{A}_j \cdot \nabla \frac{\partial \phi_i}{\partial ct} d\text{Vol} = -\int \frac{\partial \phi_i}{\partial ct} \mathbf{A}_j \cdot d\mathbf{Area} + \int \frac{\partial \phi_i}{\partial ct} \nabla \cdot \mathbf{A}_j d\text{Vol} = \int \frac{\mathbf{v}_i \cdot \hat{\mathbf{n}}_i}{cR_i^2} \mathbf{A}_j \cdot d\mathbf{Area} \rightarrow 0. \quad (115)$$

- [4] K.T. McDonald, *The Helmholtz Decomposition and the Coulomb Gauge* (Apr. 17, 2008), <http://physics.princeton.edu/~mcdonald/examples/helmholtz.pdf>
- [5] E. Henriot, *Les couples de radiation et les moments électromagnétiques*, Mem. Sci. Phys. **30**, (1936); translated as *Radiation couples and electromagnetic moments*, [http://physics.princeton.edu/~mcdonald/examples/EM/henriot\\_moments\\_36\\_english.pdf](http://physics.princeton.edu/~mcdonald/examples/EM/henriot_moments_36_english.pdf)
- [6] L. Rosenfeld, *Sur le tenseur d'impulsion-énergie*, Mem. Acad. Roy. Belg. **8**, 6, 1 (1940); translated as *On the Energy-Momentum Tensor in Selected Papers of Leon Rosenfeld*, ed. by R.S. Cohen and J.J. Stachel (D. Reidel, Dordrecht, 1979), [http://physics.princeton.edu/~mcdonald/examples/EM/rosenfeld\\_marb\\_8\\_6\\_1\\_40.pdf](http://physics.princeton.edu/~mcdonald/examples/EM/rosenfeld_marb_8_6_1_40.pdf)
- [7] L. Rosenfeld, *Sur la définition du spin d'un champ de rayonnement*, Bull. Acad. Roy. Belg. **28**, 562 (1942), [http://physics.princeton.edu/~mcdonald/examples/EM/rosenfeld\\_barb\\_28\\_568\\_42.pdf](http://physics.princeton.edu/~mcdonald/examples/EM/rosenfeld_barb_28_568_42.pdf)
- [8] F.J. Belinfante, *On the Current and the Density of the Electric Charge, the Energy, the Linear Momentum and the Angular Momentum of Arbitrary Fields*, Physica, **7**, 449 (1940), [http://physics.princeton.edu/~mcdonald/examples/EM/belinfante\\_physica\\_7\\_449\\_40.pdf](http://physics.princeton.edu/~mcdonald/examples/EM/belinfante_physica_7_449_40.pdf)
- [9] J. Serpe, *Remarques sur le Champ de Rayonnement Mesique*, Physica, **8**, 748 (1941), [http://physics.princeton.edu/~mcdonald/examples/EM/serpe\\_physica\\_8\\_748\\_41.pdf](http://physics.princeton.edu/~mcdonald/examples/EM/serpe_physica_8_748_41.pdf)
- [10] J. Humblet, *Sur le Moment D'Impulsion d'une Onde Électromagnétique*, Physica, **10**, 585 (1943), [http://physics.princeton.edu/~mcdonald/examples/EM/humblet\\_physica\\_10\\_585\\_43.pdf](http://physics.princeton.edu/~mcdonald/examples/EM/humblet_physica_10_585_43.pdf)  
[http://physics.princeton.edu/~mcdonald/examples/EM/humblet\\_physica\\_10\\_585\\_43\\_english.pdf](http://physics.princeton.edu/~mcdonald/examples/EM/humblet_physica_10_585_43_english.pdf)
- [11] C. Cohen-Tannoudji, J. Dupont-Roc and G. Grynberg, *Photons and Atoms. Introduction to Quantum Electrodynamics* (Wiley, New York, 1989), [http://physics.princeton.edu/~mcdonald/examples/EM/cohen-tannoudji\\_chap1.pdf](http://physics.princeton.edu/~mcdonald/examples/EM/cohen-tannoudji_chap1.pdf)
- [12] D.E. Soper, *Classical Field Theory* (Dover, New York, 2008).
- [13] I. Bialynicki-Birula and Z. Bialynicka-Birula, *Canonical separation of angular momentum of light into its orbital and spin parts*, J. Opt. **13**, 064014 (2011), [http://physics.princeton.edu/~mcdonald/examples/EM/bialynicka-birula\\_jo\\_13\\_064014\\_11.pdf](http://physics.princeton.edu/~mcdonald/examples/EM/bialynicka-birula_jo_13_064014_11.pdf)
- [14] A.M. Stewart, *Comparison of Different Forms for the “Spin” and “Orbital” Components of the Angular Momentum of Light*, Int. J. Opt., 728350 (2011), [http://physics.princeton.edu/~mcdonald/examples/EM/stewart\\_ijo\\_728350\\_11.pdf](http://physics.princeton.edu/~mcdonald/examples/EM/stewart_ijo_728350_11.pdf)
- [15] X.-S. Chen *et al.*, *Spin and Orbital Angular Momentum in Gauge Theories: Nucleon Spin Structure and Multipole Radiation Revisited*, Phys. Rev. Lett. **100**, 232002 (2008), [http://physics.princeton.edu/~mcdonald/examples/QED/chen\\_prl\\_100\\_232002\\_08.pdf](http://physics.princeton.edu/~mcdonald/examples/QED/chen_prl_100_232002_08.pdf)
- [16] E. Leader, *The photon angular momentum controversy: Resolution of a conflict between laser optics and particle physics*, Phys. Lett. B **756**, 303 (2016), [http://physics.princeton.edu/~mcdonald/examples/QED/leader\\_pl\\_b756\\_303\\_16.pdf](http://physics.princeton.edu/~mcdonald/examples/QED/leader_pl_b756_303_16.pdf)

- [17] B.J. Wundt and U.D. Jentschura, *Sources, Potentials and Fields in Lorenz and Coulomb Gauge: Cancellation of Instantaneous Interactions for Moving Point Charges*, Ann. Phys. (N.Y.) **327**, 1217 (2012),  
[http://physics.princeton.edu/~mcdonald/examples/EM/wundt\\_ap\\_327\\_1217\\_12.pdf](http://physics.princeton.edu/~mcdonald/examples/EM/wundt_ap_327_1217_12.pdf)  
<http://arxiv.org/abs/1110.6210>
- [18] M.V. Berry, *Optical Currents*, J. Opt. A **11**, 094001 (2009),  
[http://physics.princeton.edu/~mcdonald/examples/EM/berry\\_joa\\_11\\_094001\\_09.pdf](http://physics.princeton.edu/~mcdonald/examples/EM/berry_joa_11_094001_09.pdf)
- [19] W.H. Furry, *Examples of Momentum Distributions in the Electromagnetic Field and in Matter*, Am. J. Phys. **37**, 621 (1969),  
[http://physics.princeton.edu/~mcdonald/examples/EM/furry\\_ajp\\_37\\_621\\_69.pdf](http://physics.princeton.edu/~mcdonald/examples/EM/furry_ajp_37_621_69.pdf)
- [20] J.D. Jackson, *Relation between Interaction terms in Electromagnetic Momentum  $\int d^3\mathbf{x} \mathbf{E} \times \mathbf{B}/4\pi c$  and Maxwell's  $e\mathbf{A}(\mathbf{x},t)/c$ , and Interaction terms of the Field Lagrangian  $\mathcal{L}_{\text{em}} = \int d^3\mathbf{x} [E^2 - B^2]/8\pi$  and the Particle Interaction Lagrangian,  $\mathcal{L}_{\text{int}} = e\phi - e\mathbf{v} \cdot \mathbf{A}/c$*  (May 8, 2006), [http://physics.princeton.edu/~mcdonald/examples/EM/jackson\\_050806.pdf](http://physics.princeton.edu/~mcdonald/examples/EM/jackson_050806.pdf)
- [21] R.H. Romer, *Electromagnetic field momentum*, Am. J. Phys. **63**, 777 (1995),  
[http://physics.princeton.edu/~mcdonald/examples/EM/romer\\_ajp\\_63\\_777\\_95.pdf](http://physics.princeton.edu/~mcdonald/examples/EM/romer_ajp_63_777_95.pdf)
- [22] E.J. Konopinski, *What the electromagnetic vector potential describes*, Am. J. Phys. **46**, 499 (1978), [http://www.hep.princeton.edu/~mcdonald/examples/EM/konopinski\\_ajp\\_46\\_499\\_78.pdf](http://www.hep.princeton.edu/~mcdonald/examples/EM/konopinski_ajp_46_499_78.pdf)
- [23] D.J. Raymond, *Potential Momentum, Gauge Theory, and Electromagnetism in Introductory Physics*, (Feb. 2, 2008), [http://arxiv.org/PS\\_cache/physics/pdf/9803/9803023v1.pdf](http://arxiv.org/PS_cache/physics/pdf/9803/9803023v1.pdf)
- [24] S. Coleman and J.H. Van Vleck, *Origin of "Hidden Momentum" Forces on Magnets*, Phys. Rev. **171**, 1370 (1968),  
[http://physics.princeton.edu/~mcdonald/examples/EM/coleman\\_pr\\_171\\_1370\\_68.pdf](http://physics.princeton.edu/~mcdonald/examples/EM/coleman_pr_171_1370_68.pdf)
- [25] K.T. McDonald, *On the Definition of "Hidden" Momentum* (July 9, 2012),  
<http://physics.princeton.edu/~mcdonald/examples/hiddendef.pdf>
- [26] M. Orngotti and A. Aiello, *Surface angular momentum of light beams*, Opt. Ex. **22**, 6586 (2014), [http://physics.princeton.edu/~mcdonald/examples/EM/ornigotti\\_oe\\_22\\_6586\\_14.pdf](http://physics.princeton.edu/~mcdonald/examples/EM/ornigotti_oe_22_6586_14.pdf)
- [27] L.D. Landau and E.M. Lifshitz, *The Classical Theory of Fields*, 4th ed. (Butterworth-Heinemann, Oxford, 1975).
- [28] C.G. Darwin, *The Dynamical Motions of Charged Particles*, Phil. Mag. **39**, 537 (1920),  
[http://physics.princeton.edu/~mcdonald/examples/EM/darwin\\_pm\\_39\\_537\\_20.pdf](http://physics.princeton.edu/~mcdonald/examples/EM/darwin_pm_39_537_20.pdf)
- [29] J.D. Jackson, *Classical Electrodynamics*, 3rd ed. (Wiley, New York, 1999).
- [30] L.D. Landau and E.M. Lifshitz and L.P. Pitaevskii, *Electrodynamics of Continuous Media*, 2nd ed. (Butterworth-Heinemann, Oxford, 1984).

- [31] L. Page and N.I. Adams, Jr., *Action and Reaction Between Moving Charges*, Am. J. Phys. **13**, 141 (1945), [http://physics.princeton.edu/~mcdonald/examples/EM/page\\_ajp\\_13\\_141\\_45.pdf](http://physics.princeton.edu/~mcdonald/examples/EM/page_ajp_13_141_45.pdf)
- [32] F.J. Castro Paredes and K.T. McDonald, *A Paradox Concerning the Energy of A Dipole in a Uniform External Field* (May 3, 2004), <http://physics.princeton.edu/~mcdonald/examples/dipoleparadox.pdf>