

# “Hidden” Momentum in a River

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## 1 Problem

The flow of water in a river may not be obvious to an observer on its bank if the surface is very smooth, and that flow might be characterized as having “hidden” momentum.

The term “hidden” momentum was popularized by Shockley [1] in considerations of an electromechanical example, and essentially all subsequent use of this term has been for such examples, where one considers the system to consist of matter plus electromagnetic fields.

Recently, a definition of “hidden” momentum has been proposed by Daniel Vanzella [2] (see also [3]) which can be applied to mechanical systems as well, where a subsystem has a specified volume and can interact with the rest of the system via contact forces and/or transfer of mass/energy across its surface (which can be in motion),

$$\mathbf{P}_{\text{hidden}} \equiv \mathbf{P} - M\mathbf{v}_{\text{cm}} - \oint_{\text{boundary}} (\mathbf{x} - \mathbf{x}_{\text{cm}}) (\mathbf{p} - \rho\mathbf{v}_b) \cdot d\mathbf{Area} = - \int \frac{f^0}{c} (\mathbf{x} - \mathbf{x}_{\text{cm}}) d\text{Vol}, \quad (1)$$

where  $\mathbf{P}$  is the total momentum of the subsystem,  $M = U/c^2$  is its total “mass”,  $U$  is its total energy,  $\mathbf{x}_{\text{cm}}$  is its center of mass/energy,  $\mathbf{v}_{\text{cm}} = d\mathbf{x}_{\text{cm}}/dt$ ,  $\mathbf{p}$  is its momentum density,  $\rho = u/c^2$  is its “mass” density,  $u$  is its energy density,  $\mathbf{v}_b$  is the velocity (field) of its boundary, and

$$f^\mu = \frac{\partial T^{\mu\nu}}{\partial x^\nu}, \quad (2)$$

is the 4-force density exerted on the subsystem by the rest of the system, with  $T^{\mu\nu}$  being the stress-energy-momentum 4-tensor of the subsystem.

Does a flowing river contain “hidden” momentum according to the above definition?

## 2 Solution

### 2.1 The System Moves Along with the River

We consider a rectangular parallelepiped of length  $L$  along the  $x$ -direction of the flow, with area  $A$  perpendicular to  $\hat{\mathbf{x}}$ , and volume  $V = AL$ . The height of this parallelepiped is small enough that we can consider the pressure  $P$  of the water to be constant. We also assume that the velocity of the water is everywhere the same,  $\mathbf{v} = v\hat{\mathbf{x}}$ . The velocity of the parallelepiped is  $\mathbf{v}_0$ , also in the  $x$ -direction.

The stress tensor flowing water can be obtained via a Lorentz transformation from the its rest frame, in which quantities will be labeled with the superscript  $*$ .

The water has mass density  $\rho^*$  which includes the mass/energy of the elastic strain due to pressure  $P$ .

The stress-energy-momentum tensor in the rest frame of the water is

$$\mathbb{T}^{\star\mu\nu} = \left( \begin{array}{c|ccc} \rho^{\star}c^2 & 0 & 0 & 0 \\ \hline 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{array} \right), \quad (3)$$

The Lorentz transformation  $\mathbb{L}_x$  from the rest frame to the lab frame in which the water has velocity  $\mathbf{v} = v \hat{\mathbf{x}}$  can be expressed in tensor form as

$$\mathbb{L}_x^{\mu\nu} = \left( \begin{array}{c|ccc} \gamma & \gamma v/c & 0 & 0 \\ \hline \gamma v/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right), \quad (4)$$

where  $\gamma = 1/\sqrt{1-v^2/c^2} \approx 1 + v^2/2c^2$  when  $v \ll c$ . Hence, the energy-momentum-stress tensor in the lab frame is given by

$$\mathbb{T}^{\mu\nu} = (\mathbb{L}_x \mathbb{T}^{\star} \mathbb{L}_x)^{\mu\nu} = \left( \begin{array}{c|ccc} \gamma^2(\rho^{\star}c^2 + v^2P/c^2) & \gamma^2v(\rho^{\star}c^2 + P)/c & 0 & 0 \\ \hline \gamma^2v(\rho^{\star}c^2 + P)/c & \gamma^2(\rho^{\star}v^2 + P) & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{array} \right). \quad (5)$$

The mass density  $\rho = u/c^2 = \mathbb{T}^{00}/c^2$  in the lab frame is

$$\rho = \frac{\gamma^2}{c^2} \left( \rho^{\star}c^2 + \frac{v^2P}{c^2} \right) \approx \rho^{\star} \left( 1 + \frac{v^2}{c^2} \right), \quad (6)$$

to order  $1/c^2$ , the mass is  $M = \rho V$ , the momentum density  $\mathbf{p}$  is given by  $p_i = \mathbb{T}^{0i}/c$ ,

$$\mathbf{p} = -\frac{\gamma^2v}{c^2} (\rho^{\star}c^2 + P) \hat{\mathbf{x}} \approx \left( \rho + \frac{P}{c^2} \right) \mathbf{v}, \quad (7)$$

and the total momentum  $\mathbf{P}$  of the parallelepiped in the lab frame is

$$\mathbf{P} = \mathbf{p}V \approx \left[ \rho^{\star}V \left( 1 + \frac{v^2}{c^2} \right) + \frac{PV}{c^2} \right] \mathbf{v} = \left( M + \frac{PV}{c^2} \right) \mathbf{v}. \quad (8)$$

Then, according to the first version of definition (1), the parallelepiped has no “hidden” momentum in the lab frame,

$$\begin{aligned} \mathbf{P}_{\text{hidden}} &= \mathbf{P} - M\mathbf{v}_{\text{cm}} - \oint_{\text{boundary}} (\mathbf{x} - \mathbf{x}_{\text{cm}}) (\mathbf{p} - \rho\mathbf{v}_b) \cdot d\mathbf{Area} \\ &= \mathbf{P} - M\mathbf{v}_{\text{cm}} - \mathbf{p}V + \rho V\mathbf{v}_b = 0, \end{aligned} \quad (9)$$

since  $\mathbf{v}_{\text{cm}} = \mathbf{v}_b = \mathbf{v}_0$ .

Although the total momentum  $\mathbf{P}$  of the parallelepiped in the lab frame has two “relativistic” terms of order  $1/c^2$ , as given in eq. (8), there is no “hidden” momentum in the water whether  $\mathbf{v}_0$  is zero or not, according to definition (1).

### 2.1.1 Alternative Analysis

According to definition (1) the “hidden” momentum (of a subsystem) can also be written as

$$\mathbf{P}_{\text{hidden}} = - \int \frac{f^0}{c} (\mathbf{x} - \mathbf{x}_{\text{cm}}) d\text{Vol}, \quad (10)$$

where

$$f^\mu = \frac{\partial T^{\mu\nu}}{\partial x^\nu} \quad (11)$$

is the 4-force density exerted on the subsystem by all other subsystems, and  $T^{\mu\nu}$  is the stress-energy-momentum tensor of the subsystem (which is zero outside its bounding surface).

In the present example,  $f^0 = \partial T^{0\nu}/\partial x^\nu$  vanishes inside the parallelepiped (if we ignore the effect of gravity, as if the “river” flows in “outer space”), while having  $\delta$ -function terms at its ends. We consider that the integral in eq. (10) is taken over only the interior of the volume of the subsystem (*i.e.*, of the parallelepiped), in which case we again find  $\mathbf{P}_{\text{hidden}} = 0$ .<sup>1</sup>

## 2.2 The System is at Rest

We can also consider the system to be a parallelepiped at rest in the lab frame, in which case  $\mathbf{v}_{\text{cm}} = 0 = \mathbf{v}_b$ , and the “hidden” momentum is given by

$$\begin{aligned} \mathbf{P}_{\text{hidden}} &= \mathbf{P} - M\mathbf{v}_{\text{cm}} - \oint_{\text{boundary}} (\mathbf{x} - \mathbf{x}_{\text{cm}}) (\mathbf{p} - \rho\mathbf{v}_b) \cdot d\text{Area} \\ &= \mathbf{P} - \mathbf{p}V \approx \left( M + \frac{PV}{c^2} \right) \mathbf{v} - \left( \rho + \frac{P}{c^2} \right) V\mathbf{v} = 0. \end{aligned} \quad (12)$$

## References

- [1] W. Shockley and R.P. James, “Try Simplest Cases” Discovery of “Hidden Momentum” Forces on “Magnetic Currents”, Phys. Rev. Lett. **18**, 876 (1967), [http://physics.princeton.edu/~mcdonald/examples/EM/shockley\\_prl\\_18\\_876\\_67.pdf](http://physics.princeton.edu/~mcdonald/examples/EM/shockley_prl_18_876_67.pdf)
- [2] D. Vanzella, *Hidden momentum of (possibly open) systems* (June 29, 2012), [http://physics.princeton.edu/~mcdonald/examples/EM/vanzella\\_120629.pdf](http://physics.princeton.edu/~mcdonald/examples/EM/vanzella_120629.pdf)

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<sup>1</sup>D. Vanzella appears to argue that the subsystem consisting of the water inside the parallelepiped can be regarded as having a boundary much larger than this, in which case the boundary integral in eq. (9) vanishes and  $\mathbf{P}_{\text{hidden}} = \mathbf{P} - M\mathbf{v}_0 = M(\mathbf{v} - \mathbf{v}_0) + PV\mathbf{v}/c^2$ , while eq. (10) leads to  $\mathbf{P}_{\text{hidden}} = T^{0x} V \hat{\mathbf{x}}/c = \mathbf{p}V = \mathbf{P}$ . The inconsistency between these two results for nonzero velocity  $\mathbf{v}_0$  of the parallelepiped indicates to the author that it is not valid to take the volume and surface used in the integrals of definition (1) to be different from the volume that defines the subsystem.

- [3] K.T. McDonald, *On the Definition of “Hidden” Momentum* (July 9, 2012),  
<http://physics.princeton.edu/~mcdonald/examples/hiddendef.pdf>