

Darwin Energy Paradoxes

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1 Problem

If two electrical charges e_1 and e_2 are brought to rest (slowly) at separation R , then work

$$W = \frac{e_1 e_2}{R} \quad (1)$$

(in Gaussian units) must have been done by some external agent against the electrical force

$$\mathbf{F} = \frac{e_1 e_2}{R_{12}^2} \hat{\mathbf{n}}_{12} \quad (2)$$

on charge 1, where $R_{12} = |\mathbf{R}_{12}| = |\mathbf{r}_1 - \mathbf{r}_2|$ and $\hat{\mathbf{n}}_{12} = \mathbf{R}_{12}/R_{12}$. We then identify an (interaction) energy U_{EM} as being stored in the electric field of the system,

$$U_{\text{EM}} = \frac{e_1 e_2}{R}. \quad (3)$$

Suppose instead that the two charges have the same velocity \mathbf{v} perpendicular to their line of centers $\hat{\mathbf{n}}_{12}$, where $v \ll c$ and c is the speed of light. Then, the (Lorentz) force on charge 1 is (see, for example, the problem at the end of sec. 38 of [1] or sec. 19-4 of [2])

$$\mathbf{F} = \frac{e_1 e_2}{R_{12}^2} \hat{\mathbf{n}}_{12} \sqrt{1 - \frac{v^2}{c^2}} = -\nabla \frac{e_1 e_2}{R_{12}} \sqrt{1 - \frac{v^2}{c^2}} \approx \frac{e_1 e_2}{R_{12}^2} \hat{\mathbf{n}}_{12} \left(1 - \frac{v^2}{2c^2}\right) = -\nabla \frac{e_1 e_2}{R_{12}} \left(1 - \frac{v^2}{2c^2}\right), \quad (4)$$

This suggests that the charges can be brought to this configuration by an external agent that does work equal to the convection potential ψ ,

$$W = \psi = \frac{e_1 e_2}{R_{12}} \sqrt{1 - \frac{v^2}{c^2}} \approx \frac{e_1 e_2}{R} \left(1 - \frac{v^2}{2c^2}\right), \quad (5)$$

and that the (interaction) electromagnetic field energy of the system is also given by eq. (5).

A systematic approximation to electromagnetic fields and energies when the velocities of all charges are small compared to c , and when effects of radiation can be ignored, was given by Darwin in 1920 [3].¹ This approximation is reviewed in sec. 65 of [1], where it is implied that the Darwin Hamiltonian (energy) can be obtained from the Darwin Lagrangian,²

$$\mathcal{L} = \sum_i \frac{m_i v_i^2}{2} + \sum_i \frac{m_i v_i^4}{8c^2} - \sum_{i>j} \frac{e_i e_j}{R_{ij}} + \sum_{i>j} \frac{e_i e_j}{2c^2 R_{ij}} [\mathbf{v}_i \cdot \mathbf{v}_j + (\mathbf{v}_i \cdot \hat{\mathbf{n}}_{ij})(\mathbf{v}_j \cdot \hat{\mathbf{n}}_{ij})], \quad (6)$$

¹Darwin's analysis also ignored the magnetic moments of charge particles, which were "discovered" only in 1925 [4].

²The fourth term in eqs. (6)-(7) was given by Heaviside in art. 48 of [5].

by subtracting the terms of order $1/c^2$ from the first-order energy of the system to obtain

$$U = \sum_i \frac{m_i v_i^2}{2} - \sum_i \frac{m_i v_i^4}{8c^2} + \sum_{i>j} \frac{e_i e_j}{R_{ij}} - \sum_{i>j} \frac{e_i e_j}{2c^2 R_{ij}} [\mathbf{v}_i \cdot \mathbf{v}_j + (\mathbf{v}_i \cdot \hat{\mathbf{n}}_{ij})(\mathbf{v}_j \cdot \hat{\mathbf{n}}_{ij})]. \quad (7)$$

Then, for a pair of charges that move with the same velocity \mathbf{v} perpendicular to $\hat{\mathbf{n}}_{12}$, eq. (7) suggests that the (interaction) electromagnetic energy of the system is given by eq. (5).

Furthermore, a calculation of the interaction electromagnetic energy,

$$U_{\text{EM}} = \sum_{i>j} \int \frac{\mathbf{E}_i \cdot \mathbf{E}_j + \mathbf{B}_i \cdot \mathbf{B}_j}{4\pi} d\text{Vol}, \quad (8)$$

of a set $\{i\}$ of charges e_i using the electric and magnetic fields in the Darwin approximation appears, after integration by parts, to lead to the form

$$U_{\text{EM}} = \sum_{i>j} \frac{e_i e_j}{R_{ij}} - \sum_{i>j} \frac{e_i e_j}{2c^2 R_{ij}} [\mathbf{v}_i \cdot \mathbf{v}_j + (\mathbf{v}_i \cdot \hat{\mathbf{n}}_{ij})(\mathbf{v}_j \cdot \hat{\mathbf{n}}_{ij})]. \quad (9)$$

However, eqs. (5), (7) and (9) are all incorrect (and the convection potential is not a potential energy). Explain.

For other surprising results in the Darwin approximation, see [6].

2 Solution

2.1 Work Done to Configure a Pair of Moving Charges

2.1.1 Magnetic Fields Do No Work on Point Charges

One argument is that the external agent does work against the Lorentz force, which consists of an electric force and a magnetic force. It is well known that magnetic fields can do no work on point charges (that have no magnetic moment),³ so the external agent should be able to do work only against the electric part of the Lorentz force,

$$\mathbf{F}_E = \frac{e_1 e_2}{\sqrt{1 - v^2/c^2} R_{12}^2} \hat{\mathbf{n}}_{12} \approx \frac{e_1 e_2}{R_{12}^2} \hat{\mathbf{n}}_{12} \left(1 + \frac{v^2}{2c^2} \right). \quad (10)$$

This suggests that the charges can be brought to this configuration by an external agent that does work

$$W \approx \frac{e_1 e_2}{R} \left(1 + \frac{v^2}{2c^2} \right) = U_{\text{EM}}. \quad (11)$$

However, this argument is misleading in that it suggests the electric force between the charges, which depends on their velocities and accelerations as well as their positions, can be deduced as the gradient of a scalar potential. For the special case of constant, parallel velocities and displacements that keep $\hat{\mathbf{n}}_{12}$ constant this can still be done, but see Appendix B for a more general example.

³Magnetic forces do work on intrinsic magnetic moments of particles (see, for example, [7, 8], and in systems of charges with no magnetic moments the sum of the work done by the magnetic forces on the charges of a system (which includes work done by magnetic torques) is always zero, but the work done by the sum of the magnetic forces on the charges (which changes the translational kinetic energy) can be nonzero. For examples of the latter, see [9, 10].

2.1.2 Work Done When the Charges Are Assembled Slowly

The argument of sec. 2.1.1 can be questioned in that the external agent must provide a force equal and opposite to the total electromagnetic force, and the work done by this agent is related to that total force, and not to some piece of it. However, it is important to calculate that work carefully.

To configure the pair of charges at final separation R , one (or both) of them must have a small velocity component \mathbf{v}_n along their line of centers during this process. For charges that end up at rest after being moved together along their (fixed) line of centers, this small velocity results in a small correction to the longitudinal, purely electric force between them,⁴

$$F_n = \frac{e_1 e_2}{\sqrt{1 - v_n^2/c^2} R_{12}^2} \approx \frac{e_1 e_2}{R_{12}^2} \left(1 + \frac{v_n^2}{2c^2} \right), \quad (12)$$

So, the work required to assemble the charges is

$$W \approx \frac{e_1 e_2}{R} \left(1 + \frac{v_n^2}{2c^2} \right) \approx \frac{e_1 e_2}{R}, \quad (13)$$

where we consider v_n to be so small that terms in v_n^2/c^2 can be ignored. So, we again conclude that the electromagnetic (interaction) energy stored in the system is given by eq. (3).

For a pair of charges whose line of centers is along the y axis (with $y_2 < y_1$, so that $\hat{\mathbf{n}}_{12} = \hat{\mathbf{y}}$) which end up moving with velocity $\mathbf{v} = v \hat{\mathbf{x}}$, we suppose that they are assembled by a motion with velocities $\mathbf{v}_1 = v \hat{\mathbf{x}}$ and $\mathbf{v}_2 = v \hat{\mathbf{x}} + v_y \hat{\mathbf{y}}$ where $v_y \ll v$. Then, the electric and magnetic fields on charge 2 due to charge 1 are

$$\mathbf{E}_1 \approx -\frac{e_1}{R_{12}^2} \left(1 + \frac{v^2}{2c^2} \right) \hat{\mathbf{y}}, \quad \mathbf{B}_1 \approx -\frac{v}{c} \frac{e_1}{R_{12}^2} \left(1 + \frac{v^2}{2c^2} \right) \hat{\mathbf{z}}, \quad (14)$$

The Lorentz force on charge 2 is

$$\mathbf{F} = e_2 \left(\mathbf{E}_1 + \frac{\mathbf{v}_2}{c} \times \mathbf{B}_1 \right) \approx -\frac{e_1 e_2}{R_{12}^2} \frac{v v_y}{c^2} \hat{\mathbf{x}} - \frac{e_1 e_2}{R_{12}^2} \left(1 - \frac{v^2}{2c^2} \right) \hat{\mathbf{y}}. \quad (15)$$

During a time interval dt charge 2 moves distance $d\mathbf{s}_2 = \mathbf{v}_2 dt = (v \hat{\mathbf{x}} + v_y \hat{\mathbf{y}}) dt$, so the external agent does work⁵

$$dW = -\mathbf{F} \cdot d\mathbf{s}_2 \approx \frac{e_1 e_2}{R_{12}^2} \left(1 + \frac{v^2}{2c^2} \right) v_y dt = \frac{e_1 e_2}{R_{12}^2} \left(1 + \frac{v^2}{2c^2} \right) dy. \quad (16)$$

Although the y -component of the force is large compared to its x -component, the displacement in x is much larger than that in y , so the F_x contributes a non-negligible amount to the work. The total work done in reducing the y -separation of the charges to R is

$$W = \frac{e_1 e_2}{R} \left(1 + \frac{v^2}{2c^2} \right) = U_{\text{EM}}, \quad (17)$$

in agreement with eq. (11).

⁴The velocity v_n also results in a small magnetic field, which does not change the force in this case.

⁵The Lorentz force on charge 1 is perpendicular to \mathbf{v}_1 , so no work is done on charge 1.

2.2 Darwin's Approximation

2.2.1 Interaction Electromagnetic Energy via the Darwin Hamiltonian

The Lagrangian for a charge e of mass m (with no magnetic moment) that moves with velocity \mathbf{v} in an external electromagnetic field that is described by potentials ϕ and \mathbf{A} can be written (see, for example, sec. 16 of [1])

$$\mathcal{L} = -mc^2\sqrt{1 - v^2/c^2} - e\phi + e\frac{\mathbf{v}}{c} \cdot \mathbf{A}. \quad (18)$$

Darwin [3] worked in the Coulomb gauge, and kept term only to order v^2/c^2 . Then, the scalar and vector potentials due to a charge e that has velocity \mathbf{v} are (see sec. 65 of [1] or sec. 12.6 of [11])

$$\phi = \frac{e}{R}, \quad \mathbf{A} = \frac{e[\mathbf{v} + (\mathbf{v} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}}]}{2cR}, \quad (19)$$

where $\hat{\mathbf{n}}$ is directed from the charge to the observer, whose (present) distance is R .

Combining equations (18) and (19) for a collections of charged particles, and keeping terms only to order v^2/c^2 , we arrive at the Darwin Lagrangian,

$$\mathcal{L} = \sum_i \frac{m_i v_i^2}{2} + \sum_i \frac{m_i v_i^4}{8c^2} - \sum_{i>j} \frac{e_i e_j}{R_{ij}} + \sum_{i>j} \frac{e_i e_j}{2c^2 R_{ij}} [\mathbf{v}_i \cdot \mathbf{v}_j + (\mathbf{v}_i \cdot \hat{\mathbf{n}}_{ij})(\mathbf{v}_j \cdot \hat{\mathbf{n}}_{ij})], \quad (6)$$

where we ignore the constant sum of the rest energies of the particles.

The Lagrangian (6) does not depend explicitly on time, so the corresponding Hamiltonian,

$$\mathcal{H} = \sum_i \mathbf{p}_i \cdot \mathbf{v}_i - \mathcal{L}, \quad (20)$$

is the conserved energy of the system, where

$$\begin{aligned} \mathbf{p}_i &= \frac{\partial \mathcal{L}}{\partial \mathbf{v}_i} = m_i \mathbf{v}_i + \frac{m_i v_i^2}{2c^2} \mathbf{v}_i + \sum_{j \neq i} \frac{e_i e_j}{2c^2 R_{ij}} [\mathbf{v}_j + \hat{\mathbf{n}}_{ij}(\mathbf{v}_j \cdot \hat{\mathbf{n}}_{ij})] \\ &= m_i \mathbf{v}_i + \frac{m_i v_i^2}{2c^2} \mathbf{v}_i + \sum_{j \neq i} \frac{e_i \mathbf{A}_j(\mathbf{r}_i)}{c} = m_i \mathbf{v}_i + \frac{m_i v_i^2}{2c^2} \mathbf{v}_i + \frac{e_i \mathbf{A}_{\text{ext},i}(\mathbf{r}_i)}{c} \end{aligned} \quad (21)$$

is the canonical momentum of particle i , and $\mathbf{A}_{\text{ext},i}$ is the vector potential due to charges other than e_i . Hence, the Hamiltonian/energy is

$$U = \sum_i \frac{m_i v_i^2}{2} + \sum_i \frac{3m_i v_i^4}{8c^2} + \sum_{i>j} \frac{e_i e_j}{R_{ij}} + \sum_{i>j} \frac{e_i e_j}{2c^2 R_{ij}} [\mathbf{v}_i \cdot \mathbf{v}_j + (\mathbf{v}_i \cdot \hat{\mathbf{n}}_{ij})(\mathbf{v}_j \cdot \hat{\mathbf{n}}_{ij})], \quad (22)$$

as first derived by Darwin [3].

The part of this Hamiltonian/energy associated with electromagnetic interactions is

$$U_{\text{EM}} = \sum_{i>j} \frac{e_i e_j}{R_{ij}} + \sum_{i>j} \frac{e_i e_j}{2c^2 R_{ij}} [\mathbf{v}_i \cdot \mathbf{v}_j + (\mathbf{v}_i \cdot \hat{\mathbf{n}}_{ij})(\mathbf{v}_j \cdot \hat{\mathbf{n}}_{ij})] = \frac{1}{2} \sum_i e_i \left(\phi_{\text{ext}}(\mathbf{r}_i) + \frac{\mathbf{v}_i \cdot \mathbf{A}_{\text{ext}}(\mathbf{r}_i)}{c} \right), \quad (23)$$

where ϕ_{ext} is the electric scalar potential due to charges other than e_i . The energy (23) reduces to eqs. (11) and (17) for a pair of charges that moves with velocity \mathbf{v} perpendicular to their line of centers.⁶

How then is it possible to misinterpret sec. 65 of [1] to arrive at the erroneous eq. (7)?

This follows from misuse of a prescription given in sec. 40 of [13]: Let the Lagrangian be of the form $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}'$, where \mathcal{L}' is a small correction to the function \mathcal{L}_0 . Then, the corresponding addition \mathcal{H}' in the Hamiltonian $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}'$ is related by

$$(\mathcal{H}')_{p,q} = -(\mathcal{L}')_{\dot{q},q}. \quad (40.7)$$

This suggests that if we wish to express the Hamiltonian/energy in terms of \dot{q} and q , then $\mathcal{H}'(\dot{q}, q) = -\mathcal{L}'(\dot{q}, q)$. A more correct interpretation is that

$$\mathcal{H}'(p, q) = -\mathcal{L}'(p, q), \quad (25)$$

which indicates that the Hamiltonian is a function of the coordinates q and the corresponding canonical momenta p , so the small correction \mathcal{L}' to the Lagrangian should be expressed in terms of p and q before subtracting it from the lowest-order term $\mathcal{H}_0(p, q)$ of the Hamiltonian. Then, one could recast the Hamiltonian/energy in terms of \dot{q} and q , if desired, which can result in terms of $H_0(p, q)$ contributing to $H'(\dot{q}, q)$. That is, $H'(p, q) \neq H'(\dot{q}, q)$.

In the present example, the lowest-order Hamiltonian is, in terms of canonical momenta,

$$\mathcal{H}_0 = \sum_i \frac{p_i^2}{2m_i} + \sum_{i>j} \frac{e_i e_j}{2c^2 R_{ij}}. \quad (26)$$

To describe the small correction \mathcal{L}' to the Darwin Lagrangian (6) in terms of the \mathbf{p}_i rather than the \mathbf{v}_i it suffices to approximate the relation (21) as

$$\mathbf{v}_i = \frac{\mathbf{p}_i}{m_i} + \mathcal{O}(1/c^2). \quad (27)$$

Then, we have

$$\mathcal{L}'_{p,q} = \sum_i \frac{p_i^4}{8m_i c^2} + \sum_{i>j} \frac{e_i e_j}{2m_i m_j c^2 R_{ij}} [\mathbf{p}_i \cdot \mathbf{p}_j + (\mathbf{p}_i \cdot \hat{\mathbf{n}}_{ij})(\mathbf{p}_j \cdot \hat{\mathbf{n}}_{ij})], \quad (28)$$

from which we obtain the Darwin Hamiltonian

$$\mathcal{H} = \sum_i \frac{p_i^2}{2m_i} - \sum_i \frac{p_i^4}{8m_i^3 c^2} + \sum_{i>j} \frac{e_i e_j}{R_{ij}} - \sum_{i>j} \frac{e_i e_j}{2m_i m_j c^2 R_{ij}} [\mathbf{p}_i \cdot \mathbf{p}_j + (\mathbf{p}_i \cdot \hat{\mathbf{n}}_{ij})(\mathbf{p}_j \cdot \hat{\mathbf{n}}_{ij})], \quad (29)$$

as given in [1, 3]. To express the Hamiltonian/energy (29) in terms of velocities rather than momenta, we do not retain sufficient accuracy with the approximation (27); rather we must use eq. (21), in which case eq. (29) transforms into eq. (22).

⁶The integral form of eq. (23),

$$U_{\text{EM}} = \frac{1}{2} \int \left(\rho \phi + \frac{\mathbf{J} \cdot \mathbf{A}}{c} \right) d\text{Vol}, \quad (24)$$

shows the possibly surprising result that the electromagnetic energy in the Darwin approximation has the form of that for a system of quasistatic charge and current densities ρ and \mathbf{J} (which implies use of the Coulomb gauge; see, for example, sec. 5.16 of [11] or secs. 31 and 33 of [12]). See sec. 2.2.2 for further discussion.

2.2.2 Direct Calculation of the Interaction Electromagnetic Energy in the Darwin Approximation

The interaction electromagnetic energy associated with a set $\{i\}$ of charges e_i can be written

$$U_{\text{EM}} = \sum_{i>j} \int \frac{\mathbf{E}_i \cdot \mathbf{E}_j + \mathbf{B}_i \cdot \mathbf{B}_j}{4\pi} d\text{Vol}. \quad (8)$$

The electric and magnetic fields of a charge e at distance R from an observer follow in the Darwin approximation from the potentials (19),

$$\mathbf{E} = -\nabla\phi - \frac{\partial\mathbf{A}}{\partial t} = \frac{e}{R^2} \hat{\mathbf{n}} - \frac{e}{2c^2 R} \left[\mathbf{a} + (\mathbf{a} \cdot \hat{\mathbf{n}}) \hat{\mathbf{n}} + \frac{3(\mathbf{v} \cdot \hat{\mathbf{n}})^2 - v^2}{R} \hat{\mathbf{n}} \right] \equiv \frac{e}{R^2} \hat{\mathbf{n}} + \mathbf{E}', \quad (30)$$

$$\mathbf{B} = \nabla \times \mathbf{A} = \frac{e\mathbf{v} \times \hat{\mathbf{n}}}{cR^2}, \quad (31)$$

where $\mathbf{a} = d\mathbf{v}/dt$ is the (present) acceleration of the charge.⁷ See [14] for applications of these relations to considerations of electromagnetic momentum rather than energy.⁸

The potentials (19) are in the Coulomb gauge, so that $\nabla \cdot \mathbf{A} = 0$, and hence

$$\nabla \cdot \mathbf{E}' = 0. \quad (35)$$

The electric part of the energy (8) can be written

$$U_{\text{E}} = \sum_{i>j} e_i e_j \int \frac{\hat{\mathbf{n}}_i \cdot \hat{\mathbf{n}}_j}{4\pi R_i^2 R_j^2} d\text{Vol} + \sum_{i>j} \int \left(\frac{e_i \hat{\mathbf{n}}_i \cdot \mathbf{E}'_j}{4\pi R_i^2} + \frac{e_j \hat{\mathbf{n}}_j \cdot \mathbf{E}'_i}{4\pi R_j^2} \right) d\text{Vol} + \mathcal{O}\left(\frac{1}{c^4}\right). \quad (36)$$

It is well known (see, for example, eqs. (1)-(3) or the Appendix of [17]), that

$$\int \frac{\hat{\mathbf{n}}_i \cdot \hat{\mathbf{n}}_j}{4\pi R_i^2 R_j^2} d\text{Vol} = \frac{1}{R_{ij}}. \quad (37)$$

⁷Sec. 65 of [1] shows that in the Darwin approximation the Liénard-Wiechert potentials (Lorenz gauge) reduce to $\phi = e/R + (e/2c^2)\partial^2 R/\partial t^2$ and $\mathbf{A} = e\mathbf{v}/cR$, from which eqs. (30)-(31) also follow.

⁸The Lorentz force on a charge e' with velocity \mathbf{v}' is, in the Darwin approximation,

$$\mathbf{F} = e' \left(\mathbf{E} + \frac{\mathbf{v}'}{c} \times \mathbf{B} \right) = \frac{e'e}{R^2} \hat{\mathbf{n}} - \frac{e'e}{2c^2 R} \left[\mathbf{a} + (\mathbf{a} \cdot \hat{\mathbf{n}}) \hat{\mathbf{n}} + \frac{3(\mathbf{v} \cdot \hat{\mathbf{n}})^2 - v^2 - 2\mathbf{v}' \cdot \mathbf{v}}{R} \hat{\mathbf{n}} - \frac{2(\mathbf{v}' \cdot \hat{\mathbf{n}})\mathbf{v}}{R} \right]. \quad (32)$$

This force depends on the acceleration \mathbf{a} of the source charge e , but not on the acceleration \mathbf{a}' of the charge e' , and has noncentral terms (not along $\hat{\mathbf{n}}$).

For comparison, the (central) force law of Weber [15] (1846) is,

$$\mathbf{F}_{\text{Weber}} = \frac{e'e}{R^2} \hat{\mathbf{n}} \left[1 + \frac{1}{c^2} \left(v'^2 + v^2 - 2\mathbf{v}' \cdot \mathbf{v} - \frac{3}{2} [\hat{\mathbf{n}} \cdot (\mathbf{v}' - \mathbf{v})]^2 \right) \right] + \frac{e'e}{c^2 R} \hat{\mathbf{n}} [\hat{\mathbf{n}} \cdot (\mathbf{a}' - \mathbf{a})], \quad (33)$$

while that of Clausius [16] (1876) is,

$$\mathbf{F}_{\text{Clausius}} = \frac{e'e}{R^2} \hat{\mathbf{n}} \left(1 - \frac{\mathbf{v}' \cdot \mathbf{v}}{c^2} \right) - \frac{e'e}{c^2 R^2} [\hat{\mathbf{n}} \cdot (\mathbf{v}' - \mathbf{v})] (\mathbf{v}' - \mathbf{v}) + \frac{e'e}{c^2 R} (\mathbf{a}' - \mathbf{a}), \quad (34)$$

both of which depend on the acceleration \mathbf{a}' of the observing charge e' .

For the second integral in eq. (36), we integrate by parts to find⁹

$$\int \frac{\hat{\mathbf{n}}_i \cdot \mathbf{E}'_j}{R_i^2} d\text{Vol} = - \int \mathbf{E}'_j \cdot \nabla \left(\frac{1}{R_i} \right) d\text{Vol} = \int \frac{1}{R_i} \nabla \cdot \mathbf{E}'_j d\text{Vol} = 0. \quad (39)$$

Thus, the electric part of the interaction energy is

$$U_E = \sum_{i>j} \frac{e_i e_j}{R_{ij}}, \quad (40)$$

which holds for charges of any velocity when we work in the Coulomb gauge.

The magnetic part of the energy (8) is

$$\begin{aligned} U_M &= \sum_{i>j} \int \frac{\mathbf{B}_i \cdot \mathbf{B}_j}{4\pi} d\text{Vol} = \sum_{i>j} \int \frac{\mathbf{B}_i \cdot \nabla \times \mathbf{A}_j}{4\pi} d\text{Vol} = \sum_{i>j} \int \frac{\mathbf{A}_j \cdot \nabla \times \mathbf{B}_i}{4\pi} d\text{Vol} \\ &= \sum_{i>j} \frac{e_i \mathbf{v}_i \cdot \mathbf{A}_j(\mathbf{r}_i)}{c} = \sum_{i>j} \frac{e_i e_j}{2c^2 R_{ij}} [\mathbf{v}_i \cdot \mathbf{v}_j + (\mathbf{v}_i \cdot \hat{\mathbf{n}}_{ij})(\mathbf{v}_j \cdot \hat{\mathbf{n}}_{ij})], \end{aligned} \quad (41)$$

where we note that $\mathbf{B} \cdot \nabla \times \mathbf{A} = \epsilon_{lmn} B_l \partial A_n / \partial x_m$, so that integration by parts leads to $-\epsilon_{lmn} A_n B_m \partial B_l / \partial x_m = \epsilon_{nml} A_n \partial B_l / \partial x_m = \mathbf{A} \cdot \nabla \times \mathbf{B}$ (and not to $-\mathbf{A} \cdot \nabla \times \mathbf{B}$, which would lead to eq. (9)), and that¹⁰

$$\nabla \times \mathbf{B}_i = \frac{4\pi}{c} \mathbf{J}_i + \frac{\partial \mathbf{E}_i}{\partial ct} = \frac{4\pi e_i \mathbf{v}_i}{c} \delta(\mathbf{r} - \mathbf{r}_i) - \nabla \frac{\partial \phi_i}{\partial ct} - \frac{\partial^2 \mathbf{A}_i}{\partial (ct)^2}. \quad (43)$$

Thus, we again find the interaction electromagnetic energy $U_{\text{EM}} = U_E + U_M$ to be given by eq. (23).

A Appendix: Comment on the Electric Field in the Darwin Approximation

The part of the electric field in the Darwin approximation that depends on the acceleration is, according to eq. (30),

$$\mathbf{E}_{\mathbf{a}, \text{Darwin}} = -e \frac{\mathbf{a} + (\mathbf{a} \cdot \hat{\mathbf{n}}) \hat{\mathbf{n}}}{2c^2 R}. \quad (44)$$

⁹The surface integral resulting from the integration by parts in eq. (39) vanishes as follows:

$$\int \frac{\mathbf{E}'_j}{R_i} \cdot d\mathbf{Area} = - \int \frac{[\mathbf{a}_j + (\mathbf{a}_j \cdot \hat{\mathbf{n}}) \hat{\mathbf{n}}]}{2c^2 R_i R_j} \cdot d\mathbf{Area} + \int \frac{(\dots)}{R_i R_j^2} \cdot d\mathbf{Area} \rightarrow - \int \frac{\mathbf{a}_j \cdot \hat{\mathbf{n}}}{c^2} d\Omega = 0. \quad (38)$$

¹⁰In greater detail, the integrand $\mathbf{A}_j \cdot \nabla \times \mathbf{B}_i$ includes the term $\mathbf{A}_j \cdot \partial^2 \mathbf{A}_i / \partial (ct)^2$ which is of order $1/c^4$, while the integral of the term $\mathbf{A}_j \cdot \nabla \partial \phi_i / \partial ct$ vanishes according to

$$- \int \mathbf{A}_j \cdot \nabla \frac{\partial \phi_i}{\partial ct} d\text{Vol} = - \int \frac{\partial \phi_i}{\partial ct} \mathbf{A}_j \cdot d\mathbf{Area} + \int \frac{\partial \phi_i}{\partial ct} \nabla \cdot \mathbf{A}_j d\text{Vol} = \int \frac{\mathbf{v}_i \cdot \hat{\mathbf{n}}_i}{c R_i^2} \mathbf{A}_j \cdot d\mathbf{Area} \rightarrow 0. \quad (42)$$

This is possibly surprising in that Liénard-Wiechert electric field of an accelerating charge (see sec. 63 of [1] or sec. 14.1 of [11]), depends (explicitly) on the acceleration as

$$\mathbf{E}_{\mathbf{a},\text{L-W}} = -\frac{e}{c^2} \left[\frac{\mathbf{a} - (\mathbf{a} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}}}{R} \right]_{\text{retarded}} + \mathcal{O}\left(\frac{1}{c^3}\right). \quad (45)$$

We illustrate the compatibility of the Darwin approximation with the Liénard-Wiechert electric field for the case of a charge e that moves along the x -axis with constant acceleration a , according to $x = at^2/2$. The observer is at $x = d$ on the x -axis, so that $\hat{\mathbf{n}} = \hat{\mathbf{x}}$. Then,

$$\mathbf{E}_{\mathbf{a},\text{Darwin}} = -\frac{ea}{c^2 d} \hat{\mathbf{x}}, \quad \text{and} \quad \mathbf{E}_{\mathbf{a},\text{L-W}} = 0. \quad (46)$$

However, we should compare the total electric fields before concluding that Darwin does not agree with Liénard and Wiechert. In particular, at time $t = 0$, the Darwin approximation is that

$$\mathbf{E}_{\text{Darwin}} = \frac{e}{d^2} \left(1 - \frac{ad}{c^2} \right) \hat{\mathbf{x}}, \quad (47)$$

while the Liénard-Wiechert field is

$$\mathbf{E}_{\text{L-W}} = e \left[\frac{\hat{\mathbf{x}} - \mathbf{v}/c}{\gamma^2 R^2 (1 - \mathbf{v} \cdot \hat{\mathbf{n}}/c)^3} \right]_{\text{retarded}}. \quad (48)$$

The retarded time is $t' = t - R/c = -R/c \approx -d/c$. Then, the retarded velocity is $[v] = at' = -ad/c$ (in the $-x$ direction), the retarded Lorentz factor is $[\gamma] = 1 + \mathcal{O}(1/c^4)$, the retarded position is $[x] = at'^2/2 = ad^2/2c^2$, and the retarded distance is $[R] = d - x = d(1 - ad/2c^2)$. Using these in eq. (48), we find

$$\mathbf{E}_{\text{L-W}} \approx e \frac{1 + ad/c^2}{d^2(1 - ad/2c^2)^2(1 + ad/c^2)^3} \hat{\mathbf{x}} \approx \frac{e}{d^2} \left(1 - \frac{ad}{c^2} \right) \hat{\mathbf{x}} = \mathbf{E}_{\text{Darwin}}. \quad (49)$$

The lesson is that when converting the Liénard-Wiechert fields from retarded time to present time, the present acceleration affects all terms, whether or not they contain explicit dependence on the retarded acceleration.

B Appendix: “Classical Positronium:” $\mathbf{v}_1 = -\mathbf{v}_2 \perp \hat{\mathbf{n}}_{12}$

In the Darwin approximation, the electromagnetic energy (23) for two charges with $\mathbf{v}_1 = -\mathbf{v}_2 = \mathbf{v}$ and $\mathbf{v} \cdot \hat{\mathbf{n}}_{12} = 0$, namely $U_{\text{EM}} = e_1 e_2 / R - e_1 e_2 v^2 / 2c^2 R$, differs from that of the case $\mathbf{v}_1 = \mathbf{v}_2 = \mathbf{v}$ only in the sign of the magnetic energy $-e_1 e_2 v^2 / 2c^2 R$. This is to be expected, as the electric (Coulomb) energy is independent of the velocity, while the magnetic energy changes sign when the velocity of one of the two charges is reversed.

An example of the above situation is “classical positronium”, in which of a pair of charges $\pm e$ and mass m move in a circular orbit of radius r with common velocity v .¹¹ Then, their

¹¹In this approximation (1920), the intrinsic magnetic moments of the charges are neglected, which moments were first postulated only in 1925 [4]. For the Darwin approximation including the magnetic moments of the electron and positron, see Appendix A.2 of [18].

separation is $R = 2r$, their accelerations are

$$\dot{\mathbf{v}}_{\pm} = \mathbf{a}_{\pm} = \mp \frac{v^2}{r} \hat{\mathbf{n}} = \mp \frac{2v^2}{R} \hat{\mathbf{n}}, \quad \text{where} \quad \hat{\mathbf{n}} = \frac{\mathbf{R}}{R} = \frac{\mathbf{r}_- - \mathbf{r}_+}{|\mathbf{R}|}, \quad \text{and} \quad \dot{\mathbf{R}} = \mathbf{v}_- - \mathbf{v}_+ = 2\mathbf{v}_- \quad (\perp \hat{\mathbf{n}}) \quad (50)$$

and from eqs. (30)-(31) the fields due to the positive charge at the position of the negative charge are

$$\mathbf{E} = \frac{e}{R^2} \hat{\mathbf{n}} \left(1 - \frac{3v^2}{2c^2} \right), \quad \mathbf{B} = \frac{e\mathbf{v}_+ \times \hat{\mathbf{n}}}{cR^2} = -\frac{e\mathbf{v}_- \times \hat{\mathbf{n}}}{cR^2}. \quad (51)$$

Hence, the Lorentz force on the negative charge is

$$\mathbf{F}_- = -e \left(\mathbf{E} + \frac{\mathbf{v}_-}{c} \times \mathbf{B} \right) = -\frac{e}{R^2} \hat{\mathbf{n}} \left(1 - \frac{v^2}{2c^2} \right). \quad (52)$$

We can also obtain the force on the negative charge from the Darwin Lagrangian (6), after re-expressing it in the center-of-mass frame as

$$\mathcal{L} = mv^2 + \frac{mv^4}{4c^2} + \frac{e^2}{R} + \frac{e^2}{2c^2 R} [v^2 + (\mathbf{v}_- \cdot \hat{\mathbf{n}})^2], \quad (53)$$

using eq. (50). Then,

$$\frac{\partial \mathcal{L}}{\partial \mathbf{v}_-} = 2m\mathbf{v}_- + \frac{mv^2}{c^2} \mathbf{v}_- + \frac{e^2}{c^2 R} [\mathbf{v}_- + (\mathbf{v}_- \cdot \hat{\mathbf{n}}) \hat{\mathbf{n}}], \quad (54)$$

$$\begin{aligned} \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \mathbf{R}} &= \frac{1}{2} \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \mathbf{v}_-} = m \left(1 + \frac{v^2}{2c^2} \right) \mathbf{a}_- + \frac{e^2 v^2}{c^2 R^2} \hat{\mathbf{n}} + \frac{e^2}{c^2 R} \mathbf{a}_- = m \left(1 + \frac{v^2}{2c^2} \right) \mathbf{a}_- - \frac{e^2 v^2}{c^2 R^2} \hat{\mathbf{n}} \\ &= \frac{\partial \mathcal{L}}{\partial \mathbf{R}} = -\frac{e^2}{R^2} \hat{\mathbf{n}} \left(1 + \frac{v^2}{2c^2} \right), \end{aligned} \quad (55)$$

recalling eq. (50), and so (see also [6, 19]),

$$m \left(1 + \frac{v^2}{2c^2} \right) \mathbf{a}_- = \mathbf{F}_- = -\frac{e^2}{R^2} \hat{\mathbf{n}} \left(1 - \frac{v^2}{2c^2} \right), \quad (56)$$

in agreement with eq. (52).

Solving this for the velocity v of the charges in the orbit of diameter R , we have

$$\frac{v^2}{c^2} \approx \frac{r_0}{2R} \left(1 - \frac{r_0}{4R} \right), \quad (57)$$

where $r_0 = e^2/mc^2$ is the classical electromagnetic radius of the charges $\pm e$. Then,

$$\text{KE} \approx m \left(1 + \frac{v^2}{2c^2} \right) v^2 \approx \frac{e^2}{2R} \left(1 - \frac{r_0}{4R} \right), \quad (58)$$

and

$$U_{\text{EM}} = -\frac{e^2}{R} \left(1 - \frac{v^2}{2c^2} \right) \approx -\frac{e^2}{R} \left(1 - \frac{r_0}{4R} \right) \approx -2\text{KE}. \quad (59)$$

recalling eq. (23).

If it is desired to reduce the separation R slowly by an external force \mathbf{F}_{ext} , such that the orbit remains essentially circular at all times, then the total energy $U(R)$ of the “positronium” must be reduced, noting that

$$U(R) = \text{KE} + U_{\text{EM}} \approx -\text{KE} \approx \frac{U_{\text{EM}}}{2} \approx -mv^2 \approx -\frac{e^2}{2R} \left(1 - \frac{r_0}{4R}\right). \quad (60)$$

according to eqs. (57) and (59). Since both the angular momentum and the energy must decrease as R decreases, the external force should be tangential, not radial, with \mathbf{F}_{ext} opposite in direction to the velocities \mathbf{v}_{\pm} .¹² Then, the rate of change of separation, \dot{R} , and the rate of change of velocity, \dot{v} , are related by

$$2F_{\text{ext}}v = -\frac{dU}{dt} = -\frac{dU}{dR}\dot{R} \approx 2mv\dot{v}, \quad (61)$$

so that $\dot{v} \approx F_{\text{ext}}/m$ even though the force is in the opposite direction to the velocity. This phenomenon is called the **satellite paradox** [21, 22].

For an electromagnetic system such as “classical positronium”, the simplest way to generate a tangential external force may be place the system in a spatially uniform, but time-varying, magnetic field, which leads to the phenomenon of classical diamagnetism [23].

As the separation between the charges is reduced, we expect that the sum of the work done by the Lorentz force of each charge on the other should equal the change in the electromagnetic energy (59). Of course, the magnetic fields of the charges do no work, so we consider the work done by the electric fields (30). These fields are not derivable from a scalar potential, so we must calculate the work done along the spiral paths of the charges.

During a small time interval dt the change in radial position of a charge is $dr = \dot{r} dt$, while on the gentle spiral the path length in the azimuthal direction is

$$ds = r\dot{\phi} dt = \frac{r\dot{\phi}}{\dot{r}} dr \approx \frac{v}{\dot{r}} dr \gg dr, \quad (62)$$

where

$$\dot{\phi} \approx \frac{v}{r} \approx \sqrt{\frac{r_0}{4r^3}}c, \quad (63)$$

according to eq. (57). The velocity of the positive charge is

$$\mathbf{v}_+ = -\dot{r} \hat{\mathbf{n}} + r\dot{\phi} \hat{\phi} \approx -\dot{r} \hat{\mathbf{n}} + v \hat{\phi}, \quad (64)$$

recalling our convention that $\hat{\mathbf{n}}$ points from the positive to the negative charge. From eq. (63) we find that

$$r\ddot{\phi} \approx -\frac{3}{2}\dot{r}\dot{\phi} \approx -\frac{3}{2}\frac{\dot{r}v}{r}, \quad (65)$$

so the acceleration is

$$\mathbf{a}_+ = (r\dot{\phi}^2 - \ddot{r}) \hat{\mathbf{n}} + (r\ddot{\phi} + 2\dot{r}\dot{\phi}) \hat{\phi} \approx \frac{v^2}{r} \hat{\mathbf{n}} + \frac{\dot{r}v}{2r} \hat{\phi} = \frac{2v^2}{R} \hat{\mathbf{n}} + \frac{\dot{r}v}{R} \hat{\phi}, \quad (66)$$

¹²A transient radial force perturbs the orbit from circular to elliptical without changing the energy U or the angular momentum. See, for example, pp. 40-41 of [20].

where we neglect the term in \ddot{r} . The electric field (30) due to the positive charge at the position of the negative charge is

$$\mathbf{E} \approx \frac{e}{R^2} \hat{\mathbf{n}} - \frac{e}{2c^2 R} \left[\left(2a_n - \frac{v^2}{R} \right) \hat{\mathbf{n}} + a_\phi \hat{\phi} \right] \approx \frac{e}{R^2} \left(1 - \frac{3v^2}{2c^2} \right) \hat{\mathbf{n}} - \frac{e r v}{2c^2 R^2} \hat{\phi}. \quad (67)$$

The work done on the negative charge by this electric field as the radius of the orbit changes by dr is

$$dW_- = -eE_n dr - eE_\phi ds \approx -e \left(E_n + E_\phi \frac{v}{\dot{r}} \right) dr \approx -\frac{e^2}{R^2} \left(1 - \frac{v^2}{c^2} \right) dr \approx -\frac{e^2}{R^2} \left(1 - \frac{r_0}{2R} \right) dr, \quad (68)$$

recalling eq. (62). An equal amount of work is done on the positive charge, so the total work done by the electric fields while the radius changes by $dr = dR/2$ obeys

$$\frac{dW}{dR} \approx -\frac{e^2}{R^2} \left(1 - \frac{r_0}{2R} \right) = \frac{d}{dR} \left[\frac{e^2}{R} \left(1 - \frac{r_0}{4R} \right) \right] \approx -\frac{dU_{\text{EM}}}{dR}, \quad (69)$$

recalling eq. (59). As expected, the work done by the electric field is equal and opposite to the change in the electromagnetic energy stored in the system.

While we can write the electric force on, say, the negative charge when in a circular orbit with separation R as the gradient of a scalar potential,

$$-e\mathbf{E} \approx -\frac{e^2}{R^2} \left(1 - \frac{3v^2}{2c^2} \right) \hat{\mathbf{n}} \approx -\frac{e^2}{R^2} \left(1 - \frac{3r_0}{4R} \right) \hat{\mathbf{n}} = -\nabla_{\mathbf{R}} \left[-\frac{e^2}{R^2} \left(1 - \frac{3r_0}{8R} \right) \right], \quad (70)$$

that potential is not the electromagnetic energy U_{EM} . However, at order v^2/c^2 the electric fields in “classical positronium” are not conservative, so we should not expect that $\mathbf{F} = -\nabla U$ to hold as when $\nabla \times \mathbf{E} = 0$.

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