"Hidden" Momentum in a Leaping Beaded Chain

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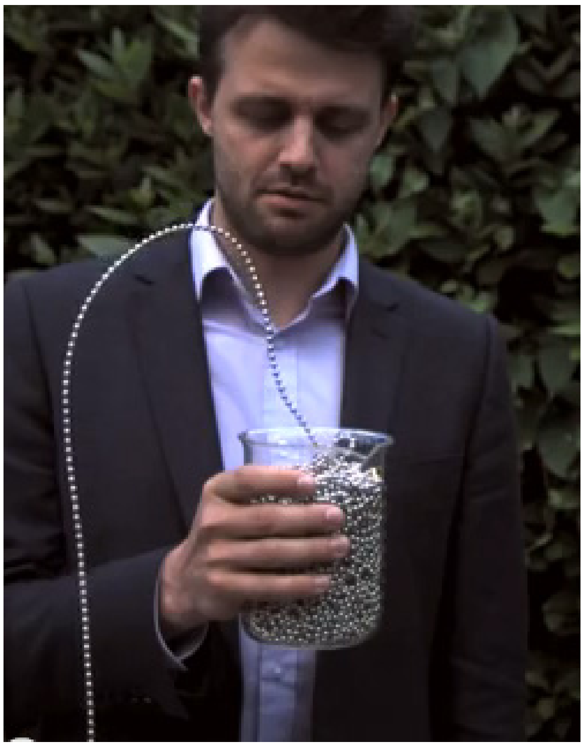
1 Problem

An impressive demonstration, in which a bead chain appears to leap upwards from a beaker, before arcing over and falling down, is given at http://www.youtube.com/watch?v=6ukMId5fIi0 See also, http://stevemould.com/siphoning-beads/




Two photos from this video are shown above.

Discuss the motion of the beaded chain, and relate it to "hidden" momentum, as defined in [1],

$$\mathbf{P}_{\rm hidden} \equiv \mathbf{P} - M\mathbf{v}_{\rm cm} - \oint_{\rm boundary} (\mathbf{x} - \mathbf{x}_{\rm cm})\,(\mathbf{p} - \rho\mathbf{v}_b) \cdot d\mathbf{Area} = -\int \frac{f^0}{c}(\mathbf{x} - \mathbf{x}_{\rm cm})\,d{\rm Vol}, \qquad (1)$$

where $\mathbf{P}$ is the total momentum of the subsystem, $M = U/c^2$ is its total "mass", U is its total energy, c is the speed of light in vacuum, $\mathbf{x}_{\rm cm}$ is its center of mass/energy, $\mathbf{v}_{\rm cm} = d\mathbf{x}_{\rm cm}/dt$, $\mathbf{p}$ is its momentum density, $\rho = u/c^2$ is its "mass" density, u is its energy density, $\mathbf{v}_b$ is the velocity (field) of its boundary, and

$$f^\mu = \frac{\partial T^{\mu\nu}}{\partial x^\nu}\,, \qquad (2)$$

is the 4-force density exerted on the subsystem by the rest of the system, with $T^{\mu\nu}$ being the stress-energy-momentum 4-tensor of the subsystem.

2 Solution

2.1 The Motion

A chain can be under tension, but not under compression. The chain at rest in the beaker cannot give energy or momentum to that part of the chain which is in the air. Rather, once the chain is in the air, the weight of the lower part of the chain creates a tension in the upper part of the chain, whose arc results in a force on the bead at rest in the beaker closest to the chain in the air. This force has both a vertical and horizontal component in general, which gives the bead an initial horizontal and vertical momentum.

The force of gravity eventually converts the upward vertical momentum into a downward momentum.

The chain in the air is observed to be essentially vertical at points below the level of the beaker, which means that the initial horizontal momentum of a bead has "disappeared" by the time that it falls vertically. It must be that the momentum of a bead is transferred to beads closer to the beaker as the bead travels in the arc.

Furthermore, once a quasi steady "flow" of the chain is achieved, the chain is nearly vertical just above the beaker. Of course, the momentum imparted by the moving chain to beads still at rest becomes part of the momentum of the moving chain, and is not "lost".

The motion must be started by pulling on the chain, typically such that it initially makes contact with the rim of the beaker, while the chain "lifts free" of the rim as the motion develops into a quasi steady state. In this quasistatic configuration a bead has very little horizontal momentum at both the beginning and the end of its motion, yet the bead moves horizontally during the motion, first picking up horizontal speed, then losing it. The horizontal momentum of the beads is essentially fixed in space, while the chain moves through this fixed pattern of stored momentum, without carrying any horizontal momentum away from the arc of the chain.

If we define the part of the chain that is in the air and above the height of the beaker as the subsystem of interest, then the center of mass of this subsystem is at rest in the lab frame (except for a tiny Zitterbewegung of amplitude $\approx$ interbead spacing). Meanwhile, the beads have horizontal motion, and nonzero associated horizontal momentum P_x. If we suppose in the first photograph on p. 1 that the x-axis points to the left, then the momentum stored in the chain points to the left: $P_x > 0$.

This is a very peculiar situation, encountered mainly in "static" electromechanical systems in which the electromagnetic fields are static and the system appears to be at rest, such that the stored electromagnetic field momentum is nonzero, while the system also possesses a "hidden" mechanical momentum equal and opposite to the electromagnetic field momentum. The unusual character of such examples was first pointed out by Shockley [2].[1]

Before assessing whether or not the present example contains "hidden" momentum, we first present a model of the motion which permits simple analytic estimates.

[1]Other "mechanical" examples with some relation to the present case, but with simpler motion, are discussed in [3, 4, 5].

2.2 Model Calculations

A feature seen in the video of the leaping chain is that there is an initial transient phase during which the height of the peak of the arc above the beaker increases, before stabilizing at, say, say height h.

Analysis of the video indicates that $h \approx 0.25$ m when the height of the beaker above the floor was $H \approx 1.2$ m, and the speed of the chain was $v \approx 6$ m/s,[2]

http://www.empiricalzeal.com/2013/07/01/the-physics-of-that-gravity-defying-chain-of-metal-beads/

Also, distance between the point on the falling chain at the same height as the beaker and the latter is roughly that same as the height h above the beaker of the top of the chain's trajectory.

2.2.1 Nonconservation of Energy

The kinetic energy $mv^2/2$ of a bead of mass m on the moving chain is less than or equal to the gravitational potential energy mgH while the bead falls to the floor at height H below the initial position of the bead,

$$v \leq \sqrt{2gH}. \tag{3}$$

While energy is conserved, to a good approximation, in the rapid acceleration of a bead from rest,[3] it appears from eq. (22) of sec. 2.2.3 that about 3/8 of the kinetic energy expected from the change in gravitational potential energy of the chain is "lost" before the chain hits the floor. This loss of kinetic energy is presumably due to inelastic collisions between the beads and the links of the bead chain, which occurs primarily in the quasivertical portion of the chain below the height of the beaker (where the observer tends to ignore possible perturbations to the falling motion).

That is, the mechanical construction of the beaded chain, which permits considerable loss of energy in bead/link collisions, is essential to the demonstration, which would not work well using, say, a string.

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[2] The link actually reports $v \approx 3$ m/s, but I suspect that there was a factor of 2 misunderstanding as to the frame rate of the camera. I will clarify this later in our own experiments.

[3] We suppose the bead spacing along the chain is L when the chain is under tension, and that a bead initially at rest is accelerated up to speed v over distance l (which may differ from L). This acceleration lasts approximately $\Delta t = l/(v/2) = 2l/v$. During this time a bead, of mass $m = \rho L$, where ρ is the (average) linear mass density of the chain, takes on momentum $\Delta P = \rho L v$ due to the tension $T = \rho g H$ in the chain at the height of the beaker (as confirmed by the more detailed analysis in sec. 2.2.3 below). Hence,

$$T = \rho g H = \frac{\Delta P}{\Delta t} = \frac{\rho L v}{2l/v} = \frac{\rho L v^2}{2l}\,, \qquad v^2 = 2gH\frac{l}{L}\,. \tag{4}$$

During this time the chain does work $W = Tl = \rho g H l$, and the gain in kinetic energy of the bead is $\Delta KE = \rho L v^2/2 = \rho g H l = W$.

Hence, all of the work done by the chain in accelerating the beads appears as the kinetic energy of the beads, to a good approximation.

A corollary of the above analysis is that the distance l over which a bead is accelerated equals the bead spacing L, assuming that $v^2 = 2gH$ as holds if the potential energy released by the falling chain appears entirely as its kinetic energy.

2.2.2 Momentum of the Chain

We model the chain in a x-y coordinate system with origin at the top of the trajectory, with the x-axis horizontal and to the left in the first figure on p. 1 (such that the velocity component v_x is always positive), and with the y-axis downwards.

The angle of the chain with respect to the positive x-axis is denoted as θ, such that

$$\tan\theta = \frac{dy}{dx} \equiv y', \qquad \cos\theta = \frac{1}{\sqrt{1+y'^2}}, \qquad \sin\theta = \frac{y'}{\sqrt{1+y'^2}}. \tag{5}$$

The chain has constant speed v everywhere along the portion that is in the air. Hence the components of the velocity and of the acceleration of the chain are

$$v_x = \frac{dx}{dt} \equiv \dot{x} = v\cos\theta = \frac{v}{\sqrt{1+y'^2}}, \qquad v_y = v\sin\theta = \frac{vy'}{\sqrt{1+y'^2}} = v_x y', \tag{6}$$

The element of the chain that occupies interval dx has length $ds = \sqrt{1+y'^2}\,dx$, and mass $dm = \rho\sqrt{1+y'^2}\,dx$, where ρ is the (average) linear mass density of the chain. The horizontal component on the momentum in the moving chain is

$$P_x = \int dm\,v_x = \int \rho\sqrt{1+y'^2}\,dx \frac{v}{\sqrt{1+y'^2}} = \rho v w, \tag{7}$$

independent of the shape of the trajectory of the chain, where w is the horizontal width of that trajectory.

As the chain pulls on beads initially at rest in the beaker, it gives them momentum. However, this momentum becomes part of that accounted for in eq. (7) and so does not change the value of P_x.

If the chain hits the floor with a nonzero horizontal velocity, then momentum would be transferred from the chain to the floor, and P_x would decrease. The empirical evidence is that the chain is rather vertical as it approaches the floor, such that transfer of momentum to the floor appears to be a rather small effect, and P_x is essentially constant at the initial value given by pulling on the chain to start its motion.

In the model given below for the steady-state trajectory of the chain, its reaches the floor with a small horizontal component v_x to its velocity, whereas the actual v_x at the floor appears to be negligible.

2.2.3 Equilibrium Trajectory of the Chain

The components of the acceleration of the chain follow from eq. (6) as

$$a_x = -\frac{vy'\dot{y}'}{(1+y'^2)^{3/2}} = -\frac{vy'y''v_x}{(1+y'^2)^{3/2}} = -\frac{v^2y'y''}{(1+y'^2)^2}, \qquad a_y = v_x\dot{y}' + a_x y' = \frac{v^2y''}{(1+y'^2)^2}, \tag{8}$$

using $\dot{y}' = y''v_x$. Note that $y'' > 0$ on the downward-arcing trajectory of the chain, since y is positive downwards.

The tension in the chain is T, and we write $T(0) = T_0$ for the tension at the top of the trajectory, where $x = y = y' = 0$.

The horizontal equation of motion for an element of the chain is

$$dm\, a_x = -\frac{\rho v^2 y' y''}{(1+y'^2)^{3/2}}\, dx = \frac{dT_x}{dx}\, dx = \frac{d}{dx}\left(\frac{T}{\sqrt{1+y'^2}}\right) dx = \frac{T'(1+y'^2) - T y' y''}{(1+y'^2)^{3/2}}\, dx, \qquad (9)$$

$$(T - \rho v^2) y' y'' = T'(1+y'^2). \qquad (10)$$

Similarly, the vertical equation of motion is (for y positive downwards)

$$\begin{aligned} dm\, a_y &= \frac{\rho v^2 y''}{(1+y'^2)^{3/2}}\, dx = dm\, g + \frac{dT_y}{dx}\, dx = \rho g \sqrt{1+y'^2}\, dx + \frac{d}{dx}\left(\frac{T y'}{\sqrt{1+y'^2}}\right) dx \\ &= \rho g \sqrt{1+y'^2}\, dx + \frac{T' y'(1+y'^2) + T' y''}{(1+y'^2)^{3/2}}\, dx, \qquad (11) \end{aligned}$$

$$(T - \rho v^2) y'' + T' y'(1+y'^2) = -\rho g (1+y'^2)^2. \qquad (12)$$

Multiplying eq (12) by y' and using eq. (10), we find

$$T' = -\rho g y', \qquad T = T_0 - \rho g y. \qquad (13)$$

That is, the tension in the chain is highest at the top of the trajectory, and, at a lower height y, is less than the maximum tension T_0 by the weight $\rho g y$ of length y of the chain. The floor is at $y = h + H$, and the tension T goes to zero there. Hence,

$$T_0 = \rho g (h + H). \qquad (14)$$

Using eq. (13) in (10), we have

$$(-\rho g y + T_0 - \rho v^2) y' y'' = -\rho g y'(1+y'^2), \qquad \frac{y' y''}{1+y'^2} = \frac{\rho g y'}{\rho g y - T_0 + \rho v^2}, \qquad (15)$$

which integrates to

$$\frac{1}{2} \ln(1+y'^2) = \ln(\rho g y - T_0 + \rho v^2) - \ln C. \qquad (16)$$

At the top of the trajectory, $y = 0 = y'$, so

$$C = \rho v^2 - T_0 = \rho[v^2 - g(h+H)], \qquad y''(y=0) = \frac{\rho g}{C} \equiv \frac{1}{R_0}, \qquad (17)$$

where R_0 is the radius of curvature of the trajectory at its apex. The constant C must be positive, which implies (recalling eq. (3)) that

$$\sqrt{2gH} > v > \sqrt{g(H+h)}. \qquad (18)$$

Then, eq. (16) leads to

$$1 + y'^2 = \left(1 + \frac{\rho g y}{C}\right)^2 = \left(1 + \frac{y}{R_0}\right)^2, \qquad y'(x > 0) = \sqrt{\left(1 + \frac{y}{R_0}\right)^2 - 1} = \sqrt{\frac{y^2}{R_0^2} + 2\frac{y}{R_0}}, \tag{19}$$

which integrates to

$$|x| = R_0 \ln \left[2 + \frac{y}{R_0} + 2\sqrt{\frac{y^2}{R_0^2} + 2\frac{y}{R_0}}\right] + D. \tag{20}$$

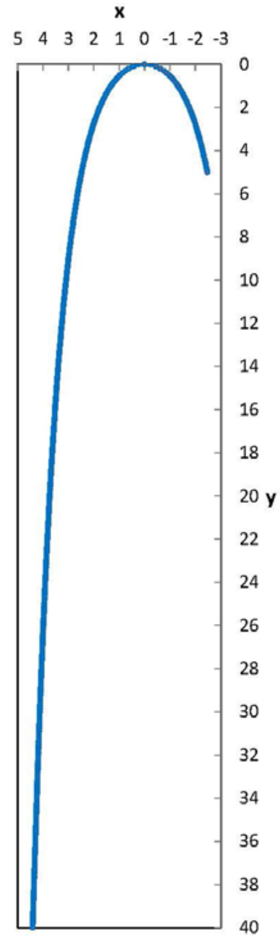
To have $x = 0 = y$ at the top of the trajectory, we need $D = -R_0 \ln 2$, and hence,

$$\frac{|x|}{R_0} = \ln \left[1 + \frac{y}{R_0} + \sqrt{\left(1 + \frac{y}{R_0}\right)^2 - 1}\right], \qquad e^{-|x|/R_0} = 1 + \frac{y}{R_0} + \sqrt{\left(1 + \frac{y}{R_0}\right)^2 - 1}. \tag{21}$$

The figure below shows a trajectory computed from eq. (21) for scale factor $R_0 = 1$. The behavior in the photograph on p. 1, in which the height h of the top of the arc above the beaker is roughly equal to the width of the curve at the height of the beaker, corresponds to a beaker position of (-2.5,5) in the scaled coordinates. That is, fitting to the shape of the upper portion of the arc implies

$$\frac{\rho g h}{C} = 5, \qquad \frac{gh}{5} = \frac{C}{\rho} = v^2 - g(h + H), \qquad v^2 = g\left(H + \frac{6h}{5}\right) \approx \frac{5gH}{4}, \qquad R_0 = \frac{h}{5}, \tag{22}$$

in "reasonable" agreement with the reported data.



2.2.4 Wave Velocity

The speed of transverse waves on the chain is given by

$$v_{\text{wave}}(y) = \sqrt{\frac{T}{\rho}} = \sqrt{g(H + h - y)}, \tag{23}$$

which is maximal at the top of the arc, $v_{\text{wave,max}} = \sqrt{g(H + h)} < v$, recalling eq. (18). However, the wave velocity at the top is quite close to the chain velocity v.[4]

A famous result is that if the wave velocity equals the speed of a rope/chain, then a perturbed waveform which propagates in the opposite direction to the motion of the chain appears to be "frozen" in space. See, for example, [7].

The author does not interpret the video as providing evidence for counterpropagating waves with "frozen" waveforms. Rather, the chain just above the beaker shows substantial transverse perturbations which are quickly damped, such that these perturbations are hardly visible after the peak of the arc. This rapid damping gives the waveform of the chain a kind of stability, rather than v_{wave} being close to the chain speed v.

[4]An amusing factoid about waves on vertical chains is that the acceleration of a pulse is $a = dv(y)/dt = (dv/dy)(dy/dt) = \pm g/2$ [6].

2.3 "Hidden" Momentum

Consideration of "hidden" momentum in electromechanical systems has led to a general definition (due to D. Vanzella [8]) of this concept, as given above in eq. (1).

In the present example, $\mathbf{v}_{\rm cm} = 0 = \mathbf{v}_{\rm boundary}$ for the subsystem consisting, say, of the arc of the chain above the beaker ($0 \le y \le h$), while that portion of the moving chain possesses nonzero horizontal momentum P_x,

$$P_x = \int_{-w/2}^{w/2} \rho\sqrt{1+y'^2}\, dx\, \gamma v_x = \gamma\rho v w. \tag{24}$$

where w is the horizontal width of the subsystem, and

$$\gamma = \frac{1}{\sqrt{1-v^2/c^2}} \approx 1 + \frac{v^2}{2c^2}\,. \tag{25}$$

Hence, $P_x - M v_{{\rm cm},x} = P_x$ is nonzero.

The boundary of the subsystem has two points, $x = \pm w/2$, $y = h$, at the ends of arc above the height of the beaker, and the boundary integral in eq. (1) has horizontal component

$$\oint_{\rm boundary} (x - x_{\rm cm})\,(\mathbf{p} - \rho\mathbf{v}_b)\cdot d\mathbf{Area} = 2\frac{w}{2}\gamma\rho v = P_x, \tag{26}$$

since $\mathbf{v}_b = 0$ at the boundary points, and $d\mathbf{Area}$ points in the negative-z direction. Thus, the "hidden" momentum according to definition (1) is zero,

$$\begin{aligned} \mathbf{P}_{\rm hidden} &= \mathbf{P} - M\mathbf{v}_{\rm cm} - \oint_{\rm boundary} (\mathbf{x} - \mathbf{x}_{\rm cm})\,(\mathbf{p} - \rho\mathbf{v}_b)\cdot d\mathbf{Area} \\ &= P_x - 0 - P_x = 0. \end{aligned} \tag{27}$$

The considerations which led to the general definition (1) showed that the "hidden" momentum can also be expressed in a more abstract form involving the time component f^0 of the 4-force which the subsystem exerts on the rest of the system. In the present example the subsystem interacts with the rest of the Universe via gravity, and via the contact forces at the two boundary points. These surface-contact forces do not contribute to the volume integral in the relation

$$\mathbf{P}_{\rm hidden} = -\int \frac{f^0}{c}(\mathbf{x} - \mathbf{x}_{\rm cm})\, d{\rm Vol}. \tag{28}$$

To evaluate f^0 via eq. (1) we note that the top row of the stress-energy tensor of the arc of the chain is

$$T^{0\mu} = (\gamma\rho[c^2 - gy], \gamma\rho v_x, \gamma\rho v_y, 0) = \gamma\rho\left(c^2 - gy, \frac{v}{\sqrt{1+y'^2}}, \frac{vy'}{\sqrt{1+y'^2}}, 0\right), \tag{29}$$

such that

$$f^0_{\rm chain} = \frac{\partial T^{0\mu}}{\partial x^\mu} = 0. \tag{30}$$

Hence, the "hidden" momentum of the arc of the chain is also zero according to eq. (28).

References

[1] K.T. McDonald, *On the Definition of "Hidden" Momentum* (July 9, 2012), http://physics.princeton.edu/~mcdonald/examples/hiddendef.pdf

[2] W. Shockley and R.P. James, *"Try Simplest Cases" Discovery of "Hidden Momentum" Forces on "Magnetic Currents"*, Phys. Rev. Lett. **18**, 876 (1967), http://physics.princeton.edu/~mcdonald/examples/EM/shockley_prl_18_876_67.pdf

[3] K.T. McDonald, *"Hidden" Momentum in a Link of a Moving Chain* (June 28, 2012), http://physics.princeton.edu/~mcdonald/examples/link.pdf

[4] K.T. McDonald, *"Hidden" Momentum in a River* (July 5, 2012), http://physics.princeton.edu/~mcdonald/examples/river.pdf

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[6] T. Foster *et al.*, *On the $g/2$ Acceleration of a Pulse in a Vertical Chain*, Phys. Teach. **51**, 394 (2013), http://physics.princeton.edu/~mcdonald/examples/mechanics/foster_pt_51_394_13.pdf

[7] C.G. Tully and K.T. McDonald, *Spinning Lasso* (Dec. 11, 2009), http://physics.princeton.edu/~mcdonald/examples/lasso.pdf

[8] D. Vanzella, *Hidden momentum of (possibly open) systems* (June 29, 2012), http://physics.princeton.edu/~mcdonald/examples/EM/vanzella_120629.pdf