Geometry Topology and Entanglement in the FQHE

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• hidden geometry of the Laughlin state
• geometro dynamics of the FQHE
• geometry and entanglement
Laughlin state

• originally introduced as a “lowest Landau level wavefunction”
  (I will explain why this is a misleading characterization)

\[ \Psi_{L}^{q}(\{r_{i}\}) = \prod_{i<j}(z_{i} - z_{j})^{q} \prod_{i} e^{-\frac{1}{2}z_{i}^{*}z_{i}} \]

• usual interpretation of \( z \) is
  \[ z = \frac{x + iy}{\sqrt{2\ell_{B}^{2}}} \]

magnetic area: \[ 2\pi\ell_{B}^{2} \]
(contains one flux quantum \( \hbar/e \))

The most striking feature for theorists is that this is holomorphic!
• Laughlin explained that his wavefunction had a holomorphic factor because it was a lowest-Landau level wavefunction.

• I will explain why the holomorphic character has a quite different origin!

• This will explain why the Laughlin state can be found in systems unrelated to lowest Landau level systems

• It will also reveal the fundamental geometric degree of freedom of the FQHE state.
standard derivation

• non-relativistic Galileian-invariant Landau levels

\[ H = \frac{|\vec{p} - e\vec{A}(\mathbf{r})|^2}{2m} = \frac{1}{2} \hbar \omega_c (a^\dagger a + aa^\dagger) \]  
(Note isotropic effective mass)

• Landau level ladder operators (in the “symmetric gauge”):

\[ a = \frac{1}{2} z + \frac{\partial}{\partial z^*} \quad a^\dagger = \frac{1}{2} z^* - \frac{\partial}{\partial z} \quad [a, a^\dagger] = 1 \]

lowest Landau level wavefunctions

\[ a \psi(\mathbf{r}) = 0 \quad \rightarrow \quad \psi(\mathbf{r}) = f(z)e^{-\frac{1}{2}z^*z} \]

holomorphic function $\times$ Gaussian
\[ \Psi^q_L(\{r_i\}) = \prod_{i<j}(z_i - z_j)^q \prod_i e^{-\frac{1}{2}z_i^* z_i} \]

- The \( q = 3 \) Laughlin state was confirmed (by numerical exact diagonalization studies) to be the essential description of the \( 1/3 \) FQHE.

- The holomorphic factor is incidentally noticed to be a cft correlator (conformal block) of the free boson cft with boson radius \( R = \sqrt{(2/q)} \).
  (why?)
• So it is known to work, but **why?** (In my opinion, this question was never satisfactorily answered)

A common rationalization:

“Laughlin’s wavefunction cleverly lowers the Coulomb correlation energy by placing its zeroes at the locations of the particles”

We will see that this is an empty statement
The physics of the FQHE in Landau levels is the physics of non-commuting “guiding centers” (quantum geometry) which cannot be described in terms of Schrödinger wavefunctions.
\[ \Psi \propto \prod_{i<j} (z_i - z_j)^q \prod_i e^{-\frac{1}{2} z_i^* z_i} \]

- \( q = 1 \) case is Slater determinant of filled lowest Landau level, uncorrelated, no topological order

- \( q > 1 \) case is highly correlated, topologically-ordered, related to a Jack polynomial

\[ \prod_{i<j} (z_i - z_j)^q = \prod_{i<j} (z_i - z_j)^{q-1} J^\alpha_\lambda (z_1, \ldots, z_N) \]

\[ \alpha = -2 \quad \text{Jack parameter} \]

\[ \lambda = \{ q(N - 1), q(N - 2), \ldots, q, 0 \} \quad \text{padded partition of N parts} \]
Jack symmetric polynomials are:

- homogeneous and symmetric
- eigenfunctions of a Laplace-Beltrami operator

\[
\alpha \sum_i (z_i \partial_{z_i})^2 J^\alpha_\lambda(z) + \sum_{i<j} (z_i - z_j)^{-1} (z_i \partial z_i - z_j \partial z_j) J^\alpha_\lambda(z) = E_\lambda(\alpha) J^\alpha_\lambda(z)
\]

- \( \alpha = -2 \) Laughlin (and quasi holes)
- \( \alpha = -3 \) Moore-Read

- \( \alpha = -k, \quad k - 1 = 3, 4, 5 \ldots \) Read-Rezayi \( \mathbb{Z}_{k-1} \) parafermion states
• single particle states \[ \psi_m \propto z^m e^{-\frac{1}{2}z^* z} \]

• radially ordered

• “root partition” of Jack:

\[ n_m(\lambda) = \sum_i \delta_{m, \lambda_i} \]

“Fermi point”

(fuzzy) boundary of circular droplet

\begin{align*}
111111111111111\mid 000000 \ldots, & \quad q = 1 \\
10010010010010\mid 000000 \ldots, & \quad q = 3 \\
\end{align*}

“not more than one particle in any group of \( q \) consecutive orbitals”

exclusion statistics (\( q > 1 \))
• “squeezing property” of Jacks

\[(z_1 - z_2)^3 = (z_1^3 z_2^0 - z_1^0 z_2^3) - 3(z_1^2 z_2^1 - z_1^1 z_2^2)\]

10010|000... 01100|000...

10010|000... \[\text{root partition } \lambda\]

01100|000... \[\text{partition } \mu \prec \lambda\]

a “squeeze”

• actual occupation of orbitals differs from root.
A Luttinger-type sum rule

\[ N = \sum_{m} n_{m}^{0} = \sum_{m} n_{m} \]

- in thermodynamic limit (map to cylinder, with circumference \( L \), then infinite plane)

\[ k = \frac{2\pi m}{L} \]

\[ \int_{-\infty}^{\infty} dk \left( n(k) - n^{0}(k) \right) = 0 \]

\[ n^{0}(k) = \begin{cases} 
1/q & k < qk_{F} \\
0 & k > qk_{F} 
\end{cases} \]
\[ n = 1 \]

\[ n = 0 \]

\[ q = 1 \]

“guiding center” occupation function

\[ \rho \rightarrow \frac{\nu}{2\pi \ell_B^2} \]

particle density

\[ q = 3 \]

“Luttinger” “guiding center” occupation function

\[ n \rightarrow 1/3 \]

\[ n \propto (3k_F - k)^2 \]

“3k_F”

\[ n = 0 \]

\[ \rho \rightarrow \frac{\nu}{2\pi \ell_B^2} \]

particle density
Finite-size Jack on cylinder (can’t get very close to -3\(k_F\))

-3\(k_F\)

(cft shows small \(k+3k_F\) behavior is quadratic)
• The occupation functions are highly structured, with generalized Fermi point singularities, and directly reflect the properties of the Jack polynomials, which are deeply related to conformal field theory.

• In contrast, the lowest-Landau electron densities derived from the interpretation of the Laughlin state as a Schroedinger wavefunction are rather smooth and featureless (WHY?)

In fact, all the non-trivial structure is present in the “guiding-center” degrees of freedom without reference to Landau level structure.
• The FQH is a correlated state of the non-commuting GUIDING CENTERS of quantized Landau orbits, obeying the algebra

\[ [R^x, R^y] = -i\ell_B^2 \]

• The classical coordinate of the electron combines this with the Landau orbit radius

\[
\begin{align*}
  r &= R + \tilde{R} \\
  [\tilde{R}^x, \tilde{R}^y] &= i\ell_B^2 \\
  [\tilde{R}^x, R^y] &= 0 \\
  [R^x, R^y] &= 0
\end{align*}
\]

• A Schroedinger wavefunction requires both degrees of freedom

\[
\Psi(r) = \langle r \mid [\psi_R \otimes \tilde{\psi}_{\tilde{R}}] \rangle
\]
• The fundamental description of FQH states is a state $|\Psi_{R_1,\ldots,R_N}\rangle$ in the many-guiding-center Hilbert space.

• To make a wavefunction, with all particles in the same Landau level, we must “dress” it with a trivial state describing Landau orbits:

$$|\Psi_{R_1,\ldots,R_N}\rangle \otimes \prod_{\otimes,i=1}^{N} |\psi_{\tilde{R}_i}\rangle$$

• We can recover $|\Psi_{R_1,\ldots,R_N}\rangle$ by “undressing” Laughlin’s wavefunction.
Schrödinger vs Heisenberg

• resolution of conflict: the two formulations of QM are equivalent:

\[ \Psi(r) = \langle r | \Psi \rangle \]

iff \( \exists |r\rangle \text{s.t.} \)

\[ \langle r | r' \rangle = 0, \quad r \neq r' \] requires an orthonormal basis in real space obeying classical locality
• classical locality (and Schrödinger-Heisenberg equivalence) fails after Landau quantization!

\[ r = R + \tilde{R} \]

\[ r = r^a e_a \]

\[ [r^a, r^b] = 0 \]

\[ \llbracket R^a, \tilde{R}^b \rrbracket = i\ell_B^2 \epsilon^{ab} \]

\[ \llbracket R^a, R^b \rrbracket = -i\ell_B^2 \epsilon^{ab} \]

\[ \llbracket \tilde{R}^a, \tilde{R}^b \rrbracket = 0 \]

\[ \ell_B^2 = \frac{\hbar}{eB} > 0 \]

\[ p_a - eA_a(r) \equiv \epsilon_{ab} \hbar \tilde{R}^a / \ell_B^2 \]
residual guiding center degrees of freedom are non-commutative

\[ r = R + \tilde{R} \]

eliminated by Landau quantization

\[ [R^a, R^b] = -i\ell_B^2 \epsilon^{ab} \]

- isomorphic to phase space, obeys uncertainty principle

- guiding centers cannot be localized within an area less than \( 2\pi\ell_B^2 \)
• The Hamiltonian governing the residual guiding-center degrees of freedom:

\[ H = \int \frac{d^2 q \ell_B^2}{2\pi} U(q) \sum_{i<j} e^{i q \cdot (R_i - R_j)} \]

\[ U(q) = \tilde{V}(q)f(q)f(-q) \]
\[ \tilde{V}(q) = \int d^2 r_{ij} V(r_{ij})e^{i q \cdot r_{ij}} \]

Fourier transformed Coulomb interaction

\[ f_n(q) = \langle \psi_n | e^{i q \cdot \tilde{R}} | \psi_n \rangle = L_n(u)e^{\frac{1}{2}u} \]

Landau level form factor

\[ u = \frac{1}{2}|q|^2\ell_B^2 \]

(\( n \) = Landau level index)

(\( \ell_B \) = Landau length)
in this limit, the state is an unentangled product of a non-trivial state of the guiding centers with a trivial state of the Landau orbits

\[ |\Psi\rangle = |\Psi_R\rangle \otimes |\Psi_{\tilde{R}}\rangle \]

FQHE is here!

depends only on \( U(q) \)

characterized by form factor \( f_n(q) \)
• In what follows, I will regard the essential FQHE state as the purely-guiding center state defined by

\[
H = \int \frac{d^2q \ell_B^2}{2\pi} U(q) \sum_{i<j} e^{iq \cdot (R_i - R_j)}
\]

\[
[R^a, R^b] = -i\ell_B^2 \epsilon^{ab}
\]

“quantum geometry*”

\[
\rho(q) = \sum_i e^{iq \cdot R}
\]

\[
[\rho(q), \rho(q')] = 2i\sin \left( \frac{1}{2} \epsilon^{ab} q_a q_b \ell_B^2 \right) \rho(q + q')
\]

*“triple” {algebra, representation, Hamiltonian} satisfies Connes’ definition
\[
[R^a, R^b] = -i\ell_B^2 \epsilon^{ab}
\]

• given a complex structure (Kähler form) one can define ladder operators

\[
\omega_a^* \omega_b = \frac{1}{2} (g_{ab} - i\epsilon_{ab})
\]

a Euclidean metric
\[\det g = 1\]

2D antisymmetric (Levi-Civita) symbol

\[
\bar{a} = (\omega_a R^a) / \ell_B
\]

\[
[\bar{a}, \bar{a}^\dagger] = 1
\]

• guiding-center “spin”:

\[
[L, \bar{a}^\dagger] = a^\dagger
\]

\[
L(g) = g_{ab} \Lambda^{ab}
\]

\[
\Lambda^{ab} = \frac{\{R^a, R^b\}}{4\ell_B^2}
\]

generators of area-preserving linear deformations of the guiding centers
• New insight: the choice of the Euclidean metric $g_{ab}$ is (so far) **arbitrary** (previous work always chose it as diag(1,1) to be congruent to the shape of the Landau orbits)

• The metric is a (hidden) variational parameter of the Laughlin **state**, and is the **fundamental physical degree of freedom** of FQHE states.

(the metric is fixed as diag(1,1) in the “Laughlin wavefunction”)

• “symmetric gauge” basis Landau level states

• basis of Landau level states with general metric

\[ |\psi_m(g)\rangle = \frac{(\bar{a}^\dagger)^m}{\sqrt{m!}} |\Psi_0(g)\rangle \]

\[ \bar{a} |\psi_0(g)\rangle = 0 \]

(central coherent state)

wavefunctions in lowest Landau level:

\[ \langle \mathbf{r} | \psi_0 \rangle \propto e^{-\frac{1}{2}z^* z} \]  
\[ \langle \mathbf{r} | \psi_0 \rangle \propto e^{\frac{1}{2} \gamma z^2} e^{-\frac{1}{2} z^* z} \]

\[ |\gamma| < 1 \]
• in the original Schroedinger/lowest Landau level language

\[ a = \frac{1}{2} z + \frac{\partial}{\partial z^*} \]
\[ a^\dagger = \frac{1}{2} z^* - \frac{\partial}{\partial z} \]

Landau levels

\[ \bar{a} = \frac{1}{2} \bar{z} + \frac{\partial}{\partial \bar{z}^*} \]
\[ \bar{a}^\dagger = \frac{1}{2} \bar{z}^* - \frac{\partial}{\partial \bar{z}} \]

Guiding centers

• general relation

\[ \bar{z} = \alpha z^* + \beta z \quad (|\alpha|^2 - |\beta|^2 = 1) \]

• original (“Laughlin wavefunction”) relation

\[ \bar{z} = z^* \quad \rightarrow \]
\[ \bar{a} e^{-\frac{1}{2} \bar{z} z^*} = a e^{-\frac{1}{2} z^* z} = 0 \]

\[ \bar{a}^\dagger f(z) e^{-\frac{1}{2} \bar{z}^* z} = z f(z) e^{-\frac{1}{2} z^* z} \]
• one can now write the Heisenberg form of the Laughlin state, liberated from any dependence on the Landau orbit geometry

\[ |\Psi^q_L(g)\rangle = \prod_{i<j} (\omega^a(R^a_i - R^a_j))^q |\Psi_0(g)\rangle \]

\[ \omega^a R^a_i |\Psi_0(g)\rangle = 0 \quad \omega^a \omega^b = \frac{1}{2} (g_{ab} - i\epsilon_{ab}) \]

• It is the exact zero-energy ground state of the “pseudopotential” model with

\[ U(q; g) = \sum_{m<q} V_m L_m (q_g^2 \ell_B^2)e^{-\frac{1}{2} q_g^2 \ell_B^2} \]

\[ V_m > 0 \quad q_g^2 \equiv g^{ab} q_a q_b \]
• coherent state basis
\[ \bar{a} | \bar{z} \rangle = \bar{z} | \bar{z} \rangle \quad | \bar{z} \rangle = e^{\bar{z} \bar{a}^\dagger - \bar{z}^* \bar{a}} | 0 \rangle \]

\[ S(\bar{z}, \bar{z}^*; \bar{z}', \bar{z}'^* ) = \langle \bar{z} | \bar{z}' \rangle = e^{\bar{z}^* \bar{z}' - \frac{1}{2} (\bar{z}'^* \bar{z}' + \bar{z}^* \bar{z})} \]

• non-null eigenstates of the overlap define an orthonormal basis
\[ \int \frac{d\bar{z}' d\bar{z}'^*}{2\pi} S(\bar{z}, \bar{z}^*; \bar{z}', \bar{z}'^* ) \psi(\bar{z}', \bar{z}'^* ) = \lambda \psi(\bar{z}, \bar{z}^* ) \]

• non-null eigenstates are degenerate with \( \lambda = 1 \)
\[ \psi(\bar{z}, \bar{z}^* ) = f(\bar{z}^* ) e^{-\frac{1}{2} \bar{z}^* \bar{z}} \]

“accidentally” coincide with lowest-Landau level wavefunctions if \( \bar{z} = z^* \)!!!
• This is the true origin of holomorphic functions in the theory of the FQHE

• NOTHING to do with lowest Landau level states, derives from overlaps between states in a non-orthogonal overcomplete basis!

• Has obvious parallels in theory of flat-band Chern insulators, where the projected lattice-site basis is non-orthogonal and overcomplete

\[ |\Psi_L^q\rangle = \prod_i \int \frac{d\bar{z}_i^* d\bar{z}_i}{2\pi} \prod_{i<j} (\bar{z}_i^* - \bar{z}_j^*)^q \prod_i e^{-\frac{1}{2} \bar{z}_i^* \bar{z}_i} |\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_N\rangle \]

“Laughlin wavefunction”

\[ \bar{a}_i |\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_N\rangle = \bar{z}_i |\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_N\rangle \]
• The metric is a physical degree of freedom that characterizes the **shape** of the correlation hole surrounding a particle in the Laughlin state
• The $1/q$ Laughlin state can be characterized as describing a “condensate” of “composite bosons” formed by “attaching” $q$ “flux quanta” (orbitals) to the particles.
• more generally, the composite boson is formed by attaching $q$ “flux quanta” to $p$ particles.

The metric describes the shape of the composite boson
1/3 Laughlin state

If the central orbital is filled, the next two are empty

The composite boson has inversion symmetry about its center

It has a “spin”

\[
\begin{array}{ccc}
\frac{1}{2} & \frac{3}{2} & \frac{5}{2} \\
1 & 0 & 0 \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3}
\end{array}
\]

\[L = \frac{1}{2}, \quad L = \frac{3}{2}\]

\[s = -1\]

the electron excludes other particles from a region containing 3 flux quanta, creating a potential well in which it is bound
\[ L = \frac{g_{ab}}{2\ell_B^2} \sum_i R_i^a R_i^b \]

\[ Q^{ab} = \int d^2r \ r^a r^b \delta \rho(r) = s\ell_B^2 g^{ab} \]

second moment of neutral composite boson charge distribution
• The composite boson behaves as a neutral particle because the Berry phase (from the disturbance of the other particles as its “exclusion zone” moves with it) cancels the Bohm-Aharonov phase.

• It behaves as a boson provided its statistical spin cancels the particle exchange factor when two composite bosons are exchanged.

\[
p \text{ particles} \quad (-1)^{pq} = (-1)^p \quad \text{fermions}
\]
\[
q \text{ orbitals} \quad (-1)^{pq} = 1 \quad \text{bosons}
\]
• The shape of the composite boson is determined by minimizing the sum of the correlation energy and the background potential energy.

• If there is no background potential, the metric is flat and the charge density is uniform.

• If there is a background potential \( g_{ab}(r) \) varies with position to give a charge density fluctuation

\[
\delta \rho(r) = e_s K(r)
\]

Gaussian curvature of metric

\[
K(r) = \frac{1}{2} \partial_a \partial_b g^{ab} + \frac{1}{8} g_{ab} \epsilon_{cd} \epsilon^{ef} \partial_e g^{ac} \partial_f g^{bd}
\]

from variation of second moment of charge distribution
from Berry phase associated with shape change

“spin”
• metric deforms (preserving $\det g = 1$) in presence of non-uniform electric field

fluid compressed by Gaussian curvature!

potential near edge

produces a dipole moment
• Hall viscosity

\[ \eta^{abcd} = \frac{eB s}{4\pi q} \frac{1}{2} (g^{ac} \epsilon^{bd} + g^{bd} \epsilon^{ac} + a \leftrightarrow b) \]

(plus a similar term from the Landau orbit degrees of freedom (Avron et al))

\[ \eta^{xxxxy} \]

current of \( \rho_x \) in \( x \)-direction

(stress force)

\[ \sigma^a_b = \epsilon_{be} \eta^{ae cf} \epsilon_{cf} \partial_c v^d \]
Hall viscosity determines a dipole moment per unit length at the edge of the fluid

- Total guiding center angular momentum of a fluid disk of $N$ elementary droplets

$$L_{gc} = \frac{1}{2\ell_B^2} g_{ab} \sum_i R^a R^b_i = \frac{1}{2} pq \bar{N}^2 + s_{gc} \bar{N}$$

- Statistical (conformal) spin
- Geometric (guiding-center) spin
- (dipole at edge)

**Momentum**

$$P_b = B\epsilon_{ab}p^b$$

**Electric dipole**
Using the occupation numbers, we can first check if they dominate (units of for FIG. 2: so on. See FIG. 2, 3, 4, 5 and 6.

In order to have the two edges not interact with each other, we need for the monomial $L_9$. The linear fit of the log-log plot gives $\log N = 1.27(\log B)$. 

If $\log B$ is too small, then the Jack becomes a charge-$p$ Laughlin state density profile: $X$ markers for $n = 0$, and $+$ markers for $n = 1$ data, and $+$ markers for $n = 2$. A color gradient from blue ($n = 0$) to red ($n = 3$) Laughlin state density profile: $X$ markers for $n = 0$, and $+$ markers for $n = 1$ and $n = 2$. For example, for $n = 1$ data, and $+$ markers for $n = 2$. A color gradient from blue ($n = 0$) to red ($n = 3$) Laughlin state density profile: $X$ markers for $n = 0$, and $+$ markers for $n = 1$ and $n = 2$. For example, for $n = 1$ data, and $+$ markers for $n = 2$.

The dipole at a segment of the edge has a momentum

$$dP_a = \frac{\hbar}{e\ell_B^2} \epsilon_{ab} dp^b$$

momentum dipole

doesn’t contribute to total momentum:

$$\oint dP_a = 0$$

it does contribute an extra term to total angular momentum:

$$\Delta L^z(g) = \hbar \oint \epsilon^{ab} g_{bc} r^c dP_a \neq 0$$

m

edge of Laughlin 1/3

orbital occupation

momentum $dP$

circular droplet

$\Delta_L^z (g) = \hbar \oint \epsilon^{ab} g_{bc} r^c dP_a \neq 0$
area (perimeter) part of entanglement is naturally measured in the boundary length per degree of freedom measured in diameters of composite bosons
Hall viscosity gives “thermally excited” momentum density on entanglement cut, relative to “vacuum”, at von Neumann temperature $T = 1$.

\[
\frac{P_{\alpha} L^a}{2\pi} = \frac{\sum_\alpha m_\alpha e^{-\xi_\alpha}}{\sum_\alpha e^{-\xi_\alpha}} = \eta_{H}^{cd} \epsilon_{ac} \epsilon_{bd} \frac{L^a L^b}{2\pi \ell_B^2} + \frac{1}{24} (\tilde{c} - \nu) - \hbar
\]

signed conformal anomaly (chiral stress-energy anomaly)

“CASIMIR MOMENTUM”

(NOT “real-space cut” which requires the Landau orbit degrees of freedom and their form factor to be included)

virasoro level of sector

chiral anomaly
Laughlin $\nu=1/3$ state: topologically conserved “chiral central charge” is explicitly seen to be $\tilde{c} = 1$

$$\frac{1}{24} (\tilde{c} - \nu) = \frac{1}{24} \left( 1 - \frac{1}{3} \right) = \frac{1}{36}$$

MPS calculation: (“plevel” is Virasoro level at which auxiliary space is truncated, which causes errors at large L)
• other consequences of geometry

• long-wavelength of GMP mode is “graviton”

• $q^4$ behavior of guiding-center structure factor is due to zero-point fluctuations of the metric (components do not commute, determinant is Casimir)

• Conformal field theory has a fixed metric; geometrodynamics is like extension of special relativity to GR!
unfortunately, long-wavelength limit of “graviton” collective mode is hidden in “two-roton continuum”

\[ \Delta E \]

goes into continuum

numerical finite-size diagonalization

“roton”

Laughlin (inversion and translation invariant)

\[ \nu = \frac{1}{3} \]

Momentum \( \propto k \ell_B \)

gap \iff \text{incompressibility}

Gap for tangential electric polarization (no dielectric screening)

single-mode approx.

\[ E(q) s(q) \leq \frac{1}{2} G^{abcd} q_a q_b q_c q_d \ell_B^2. \]
Geometric action

\( S = \int d^3 x L_0 - H_0 \)

\( L_0 = \frac{1}{4\pi pq \hbar} \epsilon^{\mu \nu \lambda} (peA_\mu - s\Omega^g_\mu) \partial_\nu (peA_\lambda - \hbar s\Omega^g_\lambda) \)

(reduces to electromagnetic Chern-Simons action when \( s = 0 \) (integer QHE))

\( H_0 = J^0 U(J^0 g) \quad J^0 = \frac{1}{2\pi pq \hbar} (peB - \hbar s J^0_g) \)

Gaussian curvature

\( J^\mu = \epsilon^{\mu \nu \lambda} \partial_\nu \Omega^g_\lambda \)
**Geometric distortion energy**

\[ \mathcal{H}_0 = (\det G)^{1/2} U(G) = J^0 U(J^0 g) \]

**correlation energy density**

**geometric chemical potential**

(of composite bosons)

\[ \mu_g = U(G) + G_{ab} \frac{\partial U}{\partial G_{ab}} \]

**shear-stress tensor**

(traceless)

\[ \sigma^a_b = 2G_{bc} \frac{\partial U}{\partial G_{ac}} - \delta^a_b G_{cd} \frac{\partial U}{\partial G_{cd}} \]

\[ \sigma^a_a = 0 \]

\[ \sigma^a_c(x) \epsilon^{bc} = \sigma^{bc}(x) \epsilon^{ac} \]

\[ \sigma^a_c(x) g^{bc}(x) = \sigma^{bc}(x) g^{ac}(x) \quad \text{both expressions are symmetric in } a \leftrightarrow b \]

**Stress tensor is traceless because the gapped quantum incompressible fluid does not transmit pressure**

(unlike incompressible limit of classical incompressible fluid, which has speed of sound \( v_s \rightarrow \infty \))
Euler equation

- action is minimized by Hall viscosity condition

\[ J^0 \sigma^a_b (G) = \eta^{ac}_{bd} (G) \nabla^g_c J^d \]

Traceless stress-tensor

Hall viscosity

\[ \eta^{ac}_{bd} (G) = \frac{1}{2} \hbar s \epsilon_{be} \epsilon_{df} J^0 \Gamma^{aefc}_{H} (g) \]

\[ \Gamma^{abcd}_{H} (g) = \frac{1}{2} (\epsilon^{ac} g^{bd} + \epsilon^{ad} g^{bc} + \epsilon^{bc} g^{ad} + \epsilon^{bd} g^{ac}) \]

\[ \eta^{ab}_{ac} = \eta^{ba}_{ca} = 0 \] incompressible

composite boson current

covariant spatial gradient of \( J^a = J^0 u^a \)

fluid flow-velocity

dissipationless
- composite boson current

\[ J^0 = \frac{1}{2\pi pq\hbar} \left( \epsilon^{ab} peB - \hbar s J^0_g \right) \]

\[ J^a = \frac{1}{2\pi pq\hbar} \left( \epsilon^{ab} \left( peE_b - \partial_b \mu_g \right) - \hbar s J^a_g \right) \]

responds to gradient of geometric chemical potential as well as electric field

\[ pe E_a J^a \neq 0 \]

Energy flow from electromagnetic field to FQH fluid

\[ pe \left( J^0 E_a + \epsilon_{ab} J^a B \right) \neq 0 \]

tangential momentum flow from electromagnetic field to FQH fluid
Action gives gapped spin-2 (graviton-like) collective mode that coincides at long wavelengths with the “single-mode approximation” of Girvin-MacDonald and Platzman.

charge fluctuations relative to the background charge density fixed by the magnetic flux are given by the Gaussian curvature

\[
J^0_g = -\frac{1}{2} \partial_a \partial_b g^{ab} + \frac{1}{8} g_{ac} \epsilon_{bd} \epsilon^{ef} (\partial_e g^{ab}) (\partial_f g^{cd})
\]

\[
\delta J^0_e = \frac{e^* s}{2\pi} J^0_g
\]

zero-point fluctuations of gaussian curvature give quantitatively correct \(O(q^4)\) structure factor
• near edges: fluid is compressed at edges by creating Gaussian curvature

\[ \delta J_0^e = \frac{e^* s}{2\pi} J_0^g \]

fluid density fixed by flux density

\[ g = \begin{pmatrix} \alpha(x) & 0 \\ 0 & \frac{1}{\alpha(x)} \end{pmatrix} \]

\[ J_0^g = -\frac{1}{2} \frac{d^2}{dx^2} \frac{1}{\alpha(x)} \]

For larger \( s \), fluid becomes more compressible (less distortion needed for a given density change)
initial numerical study of Laughlin 1/3 on cylinder with edges
(with Zlatko Papic and Sonika Johri)

predicted

$\Delta n$

edge effects?

Variation of Gaussian curvature of pseudopotentials
SUMMARY

• New collective geometric degree of freedom leads to a description of the origin of incompressibility in FQHE in a continuum “geometric field theory”

• many new relations: guiding-center spin characterizes coupling to Gaussian curvature of intrinsic metric, stress in fluid, guiding-center structure-factors, etc.

http://wwwphy.princeton.edu/~haldane

Can be also be accessed through Princeton University Physics Dept home page (look for Research:condensed matter theory)

also see arXiv (search for author=haldane)