Geometry of the fractional quantum Hall effect

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- The fractional QHE has a local collective degree of freedom that is a dynamical 2D spatial metric tensor
- Electrons couple to the Gaussian curvature of this metric with a topological quantum number, the "guiding center spin"


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• elementary unit of the FQHE fluid with $v = \frac{p}{q}$ is a "composite boson" of $p$ electrons that exclude other electrons from a region with $q$ London ($\frac{h}{e}$) flux quanta.

$\nu = \frac{1}{3}$ Laughlin

$p = 1, q = 3$

$\frac{1}{3}$ Laughlin

(with different shape)

$\nu = \frac{1}{3}$

$p = 2, q = 5$

$\frac{2}{5}$ Hierarchy/Jain

The rule formerly known as "odd-denominator", (but Moore-Read has $p = 2, q = 4$)

Statistical selection rule:

$$(-1)^p \times (-1)^{pq} = +1$$

- Exchange of $p$ fermions
- Berry phase (exchange of "exclusion zones")

Composite is a boson
• The **shape** of the “composite boson” is a dynamical local variable defined by a **unimodular 2D spatial metric** $g_{ab}(r,t)$.

• The **orbitals** inside the “composite boson” centered at $r$ are defined by $L(r)|\Psi_m(r)\rangle = (m + \frac{1}{2})|\Psi_m(r)\rangle$.

$$L(r) = \frac{g_{ab}(r)}{2\ell_B^2} \delta R^a \delta R^b$$

$L$ includes “zero-point” contribution.

• The **non-commuting** “Landau-orbit guiding center coordinates” obey

$$[\delta R^a, \delta R^b] = -i\ell_B^2 \epsilon^{ab}$$

$\delta R = \delta R^a e_a$

- $\delta R$ is the displacement of the “guiding center” from $r$.

Guiding center orbitals are concentric ellipses centered at $r$. 

$|eB|$
• The metric (shape of the composite boson) has a preferred shape that minimizes the correlation energy, but fluctuates around that shape.

• The zero-point fluctuations of the metric are seen as the $O(q^4)$ behavior of the “guiding-center structure factor” (Girvin et al., (GMP), 1985).

• The metric has a companion “guiding center spin” that is topologically quantized in incompressible states. Together they define a symmetric rank-2 tensor guiding center “Hall viscosity”.

\[
\eta^a_b = B \gamma^a_b = \frac{eB}{4\pi q} sg^{ab}
\]

Units of viscosity: guiding-center spin

\[
\gamma_H^{abcd} = \frac{1}{2} \left( \epsilon^{ac} \gamma_H^{bd} + \epsilon^{ad} \gamma_H^{bc} + \alpha \leftrightarrow b \right)
\]

\[
\sigma^a_b = \epsilon_{bc} \gamma_H^{acde} \partial_d E_e
\]

(Traceless) stress tensor: electric-field gradient

Note: stress is fundamentally a mixed-index tensor

Upper or lower index: displacement or derivative

Note: $\gamma_H^{ab} = \gamma_H^{ba}$ is even under time-reversal.
The **guiding-center spin** $s$: a new fundamental emergent topological property of the incompressible FQHE state.

For $N$ composite bosons

\[ L = \frac{1}{2} pq \bar{N}^2 + s \bar{N} \]
The guiding center spin (as given here, and unlike the old "shift" concept) has the properties:

- It is odd under particle-hole transformations of a Landau level
- It vanishes for empty and filled Levels
- It is negative for "electron-type" FQHE fluids, and positive for hole-type fluids

Example:

Moore-Read "Pfaffian"
\[ p = 2, q = 4, s = -2 \]

Moore-Read "anti-Pfaffian"
\[ p = 2, q = 4, s = +2 \]
- Fundamental significance of the guiding-center spin:
  - Gauge coupling of composite boson is to difference of charge times magnetic flux density and guiding-center spin times Gaussian curvature!

\[ \rho_e = \frac{1}{q} \left( p \frac{eB}{2\pi \hbar} - 2s \frac{K(g)}{4\pi} \right) \]

- Magnetic flux density in units of London flux quantum
- Gaussian curvature of guiding-center metric in natural units of \( \frac{4\pi}{q} \)

\[ \int d^2r \left( p \frac{eB}{2\pi \hbar} - 2s \frac{K(g)}{4\pi} \right) = \]

The count of independent quantum states of the composite boson (an integer, if surface is compact!)

- (note analogy to coupling to Berry curvature of spin in quantum Hall ferromagnets!)

- 2s is topologically-quantized as an integer by Gauss-Bonnet theorem
charge-density fluctuations relative to uniform density fixed by magnetic flux:

\[ \delta \rho_e = -\frac{2s}{4\pi q} K \]

\[ K = -\frac{1}{2} \partial_a \partial_b g^{ab} + \frac{1}{8} g_{ab} \epsilon^{cd} \epsilon_{ef}(\partial_c g^{ae})(\partial_d g^{bf}) \]

Brioschi formula for Gaussian curvature

This is why \( \lim_{q \to 0} \delta \rho(q) |\Psi_0\rangle \sim O(q^2) \) gives correct coefficient of \( O(q^4) \) leading behavior of \( S(q) \propto \langle \Psi |\{\rho(q), \rho(-q)\}|\Psi\rangle \)
• Origin of FQHE incompressibility is analogous to origin of **Mott-Hubbard gap** in lattice systems.

• There is an energy gap for putting an **extra particle** in a quantized region that is **already occupied**.

• **On the lattice** the “quantized region” is an atomic orbital with a fixed shape.

• **In the FQHE** only the **area** of the “quantized region” is fixed. The **shape** must adjust to minimize the correlation energy.
The Laughlin state has a true variational parameter, its “guiding center metric”: why was this missed for almost 30 years?

Were theorists mesmerized by the mathematical beauty of lowest-Landau-level wavefunctions?

\[ a = \frac{1}{2} z + \partial_z^* \quad \text{Landau-index lowering operator} \]

\[ a^\dagger = \frac{1}{2} z^* - \partial_z \quad \text{Landau-index raising operator} \]

\[ [a, a^\dagger] = 1 \quad \text{form of lowest Landau-level wavefunctions:} \]

\[ a \Psi(r) = 0 \]

\[ \Psi(r) = f(z) e^{-\frac{1}{2} z^* z} \quad \text{holomorphic} \]

\[ \Psi_{q/L}^I(\{r_i\}) \propto \prod_{i<j}(z_i - z_j)^q \prod_i e^{-\frac{1}{2} z_i^* z_i} \quad \text{where's the geometry hiding?} \]

\[ z = \frac{x + iy}{\sqrt{2 \ell_B}} \]
\[ \Psi_{\mathcal{L}}^{q}(\{r_i\}) \propto \prod_{i<j} (z_i - z_j)^q \prod_i e^{-\frac{1}{2}z_i^* z_i} \]

- Commentaries on the Laughlin wavefunction generally emphasize that it "cleverly" puts all its zeroes (as a function of any one particle coordinate) at the locations of the other particles.

- This is said to efficiently reduce the correlation energy and accounts for the "variational" success of a model wavefunction without any variable parameter.

What is wrong with this conventional argument?
• fractionally-filled lowest-Landau-level physics can be completely described (only) in terms of the non-commuting guiding centers.

\[ H = \int \frac{d^2 q \ell_B^2}{4\pi} \tilde{\nu}(q) \rho(q) \rho(-q) \]

Girvin, MacDonald and Platzman (1985)

\[
[\rho(q), \rho(q')] = 2i \sin \left(\frac{1}{2} q \times q' \ell_B^2\right) \rho(q + q')
\]

Lie algebra of generators of area-preserving diffeomorphisms of the “quantum plane”

• Because locality is lost ("quantum fuzziness") Schrödinger wavefunctions in real space cannot describe the physics, only Heisenberg states in Hilbert space are valid!
Because of this (in the context of physics projected to a Landau level) **NO PHYSICAL MEANING WHATSOEVER** can be attached to wavefunctions, or the locations of their zeroes!!!!!

The “correct” description of the Laughlin state (**not “wavefunction”**) must be given in **Hilbert space**

This turns out to be the “pseudopotential” description (FDMH, 1983), which on close inspection contains a metric as a free parameter undetermined by the shape ($z \propto x + iy$) of the Landau orbit
metric-dependent pseudopotential interaction for which $1/m$ Laughlin state is unique exact ground state

$$H(g) = \int \frac{d^2 q \ell_B^2}{4\pi} \tilde{v}(q; g) \rho(q) \rho(-q)$$

$$\tilde{v}(q; g) = \sum_{m'=0}^{m-1} V_{m'} L_{m'}(q_g^2 \ell_B^2) e^{-\frac{1}{2} q_g^2 \ell_B^2}$$

$${q_g}^2 \equiv g^{ab} q_a q_b$$

unimodular
inverse metric

Pseudopotentials $V_{m'} > 0$
Laguerre Polynomials

- Heisenberg Laughlin state (in guiding-center Hilbert space)

$$|\Psi^{1/m}_L(g)\rangle \propto \prod_{i<j} (\bar{a}^\dagger_i(g) - \bar{a}^\dagger_j(g))^m |\Psi_0(g)\rangle \quad \bar{a}_i(g)|\Psi_0(g)\rangle = 0$$

$$[L(g), \bar{a}^\dagger(g)] = \bar{a}^\dagger(g) \quad \text{guiding-center } L(g)\text{-raising operator}$$
• Lowest Landau level (in Landau-orbit Hilbert space)
  \[ a_i |\Psi_0\rangle = 0 \]

• To make a Schrödinger wavefunction we need to make a direct (unentangled) product of the Heisenberg states in the two Hilbert spaces:
  \[ \Psi^{1/m}_L (r_i; g) = \langle \{ r_i \} | \left( |\Psi_0\rangle \otimes |\Psi^{1/m}_L (g)\rangle \right) \]

• Zeroes only coincide with other particles if
  \[ \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]

SUMMARY

• New collective geometric degree of freedom leads to a description of the origin of incompressibility in FQHE in a continuum “geometric field theory”

• many new relations: guiding-center spin characterizes coupling to Gaussian, stress in fluid, guiding center structure factors, etc.

• This talk and other related talks are available at

http://wwwphy.princeton.edu/~haldane

Can be also be accessed through Princeton University Physics Dept home page (look for Research:condensed matter theory)

also see arXiv (search for author=haldane)