Geometry and Incompressibility in the Fractional Quantum Hall effect

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- A new viewpoint on the Laughlin State leads to a quantitative description of incompressibility in the FQHE
- A marriage of Chern-Simons topological field theory with "quantum geometry"


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Geometry of the Fractional QHE

- For at least 20 years, most “theoretical” work (as opposed to numerical simulation) has been “topological” in character.

- Finite-size exact diagonalization (up to 20 particles) confirms that microscopic Hamiltonians exhibit incompressibility.

- Energy-scale analysis shows the gap must be of order

\[ \Delta E \sim \frac{e^2}{4\pi \varepsilon_0 \varepsilon \ell_B}, \quad \ell_B = \sqrt{\frac{\hbar}{|eB|}} \]
• A part of the phenomenology of the FQHE is successfully described by Chern-Simons Topological Quantum Field theory.

• This models the idea of “flux attachment”:

\[ S = \int d^2 r dt \, A_\mu \, J^\mu - \frac{\hbar K_{ij}}{4\pi} \epsilon^{\mu\nu\lambda} a^i_\mu \partial_\nu a^j_\lambda \]

\[ J^\mu = \frac{q_i e}{2\pi} \epsilon^{\mu\nu\lambda} \partial_\nu a^i_\lambda \]

\[ \sigma^H = \frac{e^2}{2\pi\hbar} (K^{-1})^{ij} q_i q_j \]

• Note that TQFT does \textbf{not} contain any energy scales or lengthscales.

• It describes fractional charges and statistics of quasiparticles and forces the existence of edge modes.
• Chern-Simons theory takes the existence of incompressibility as given, and does not describe it.

• Like all TQFT's its action is linear in time-derivatives of fields, and it has an elegantly simple Hamiltonian: \( H = 0 \)

Various plausible but heuristic Hamiltonians have been proposed to describe incompressibility based on:

• Ginzburg-Landau composite boson model (Galilean-invariant superfluid with a Chern-Simons term)

• Composite fermions formed by flux attachment that fill implicitly Galilean-invariant “effective Landau levels”

These pictures have existed for over 20 years, but not been derived from the microscopic physics, and do not make any contact with certain fundamental properties of incompressibility discovered in 1985 by Girvin Macdonald and Platzman,
The description of the FQHE in a partially-filled Landau level should be completely derivable from (only) the microscopic Hamiltonian

$$H = \int \frac{d^2 q \ell_B^2}{2\pi} \sum_{q \neq 0} v(q) \sum_{i < j} e^{iq \cdot (R_i - R_j)}$$

$$\int \frac{d^2 q \ell_B^2}{2\pi} v(q) = \varepsilon_0$$

finite

$$[R^a, R^b] = -i \ell_B^2 \epsilon^{ab} \delta_{ij}$$

"Non-commutative geometry"

This model does not have the classical property of locality, and its states have no Schrödinger wavefunction representation in coordinate space, just a Heisenberg one in Hilbert space!
How can I claim that a wavefunction description is not appropriate, given that a large fraction of successful work on FQHE is based on wavefunctions?

\[ r^a = R^a + \hbar^{-1} \epsilon^{ab} \pi_b \ell_B^2 \]

\[ \pi_a = p_a - eA_a(r) \]

\[ [r^a, r^b] = 0 \]

Need to recombine guiding center with dynamical momentum to make the classical coordinate.

Can’t be done if only the guiding centers should be referenced!
Laughlin state

- commonly characterized as a wavefunction:

\[ \psi = \prod_{i<j} (z_i - z_j)^3 \prod_i e^{-\frac{1}{2} z_i^* z_i} \]

\[ a_i \psi = 0 \quad \text{lowest Landau level} \]

Usual characterization: “Laughlin state places all zeroes of the wavefunction as a function of \( z_i \) at the location of the other particles”

\[ a_i^\dagger = \frac{1}{2} z_i^* - \frac{\partial}{\partial z_i} \quad \text{Landau level raising operator} \]
According to my claim that the physics of the FQHE should entirely be found in guiding center physics, this conventional characterization of the Laughlin state is **nothing to do with** why it is successful!!

In addition it seems that the Laughlin state is related to the shape of the Landau orbit, which is $|z| = \text{constant}$.

This also turns out to be a fundamental misconception about the Laughlin state.
• The integer QHE is controlled only by the geometry of the Landau orbits.

• The FQHE involves a second geometry: that of the Coulomb equipotentials of a point charge on the 2D surface. This has no fundamental relation to Landau orbit shape.

Two distinct ways to define the “circle”
I propose that the “correct” interpretation of the $1/q$ Laughlin state is based on the two-particle “pseudopotentials”

$$\int \frac{d^2q \ell_B^2}{2\pi} V_m(q_g) \sum_{i<j} e^{i q \cdot (R_i - R_j)} |\Psi^{(q)}_L(\vec{g})\rangle = 0, \quad m < q.$$ 

$$V_m(q_g) = L_m(q_g^2 \ell_B^2) e^{-\frac{1}{2} q_g^2 \ell_B}$$

This reveals that the Laughlin state is a continuous function of the “guiding-center metric” which fixes the shape of its guiding-center correlation hole.

Unimodular (det = 1) inverse 2D metric with Euclidean signature
This alternate definition reveals that there is not a single \( 1/q \) “Laughlin state”, but a two-parameter family of states with a variable geometry. This is a pure guiding-center definition.

\[
\Psi_L^{(q)}(\{r_i\}; g, \bar{g}) = \left\langle \{r_i\} \middle| \left[ |\Psi_0(g)\rangle \otimes |\Psi_L^{(q)}(\bar{g})\rangle \right] \right\rangle
\]

- **basis of simultaneous eigenstates of the commuting classical coordinates**
- **Landau-orbit coherent state** \( a_i(g) |\Psi_0(g)\rangle \)
- **guiding-center metric** \( m_{ab} = mg_{ab} \) (Galileian metric)

**zeroes of wavefunction only coincide with particles if** \( g = \bar{g} \)
• The fact that the Laughlin state depends continuously on a 2D spatial metric was apparently previously unnoticed.

• If the Coulomb point-charge equipotentials are not congruent with Landau orbits, the metric is a true variational parameter which must be chosen to minimize the correlation energy.

• More profoundly, we can identify the metric tensor $g_{ab}(r,t)$ as a collective field of the FQHE state.
relation between guiding-center and Landau-orbit coordinates

- introduce guiding center operators

<table>
<thead>
<tr>
<th>Guiding centers</th>
<th>Landau orbits</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \overline{a}_i^\dagger = \frac{1}{2} \overline{z}_i^\ast - \frac{\partial}{\partial \overline{z}_i} )</td>
<td>( a_i^\dagger = \frac{1}{2} z_i^\ast - \frac{\partial}{\partial z_i} )</td>
</tr>
<tr>
<td>( \overline{a}_i = \frac{1}{2} \overline{z}_i + \frac{\partial}{\partial \overline{z}_i^\ast} )</td>
<td>( a_i = \frac{1}{2} z_i + \frac{\partial}{\partial z_i^\ast} )</td>
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</tbody>
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Conventional choice: \( \overline{z}_i = z_i^\ast \)
• The Laughlin state can be rewritten as

\[ \Psi_L = \prod_{i<j} \left( \bar{a}_i^\dagger - \bar{a}_j^\dagger \right)^q \Psi_0 \quad \Psi_0 = \prod_i e^{-\frac{1}{2} z_i \bar{z}_i} \]

\[ a_i \Psi_0 = 0 \quad \bar{a}_i \Psi_0 = 0 \]

• If, for any i,j this state is expanded in eigenstates of relative guiding-center angular momentum

\[ L_{ij} = \frac{1}{2} (\bar{a}_i^\dagger - \bar{a}_j^\dagger)(\bar{a}_i - \bar{a}_j) = 0, 1, 2, \ldots \]

No pair of particles has \( L_{ij} < q \)

This is a fundamentally “Heisenberg” description of the Laughlin state formulated completely in terms of guiding-center physics, unlike the previous “Schroedinger” one
\( \bar{z} \text{ does not have to be } z^{\ast} ! \)

- The guiding center geometry is fixed by the geometry of Coulomb equipotentials of a point charge, which do not have to be congruent to the shape of the Landau orbits
- The second (but not the original) definition of the Laughlin state remains valid if

\[
\begin{pmatrix}
\bar{z} \\
\bar{z}^{\ast}
\end{pmatrix}
= \begin{pmatrix}
\alpha & \beta \\
\beta^{\ast} & \alpha^{\ast}
\end{pmatrix}
\begin{pmatrix}
z^{\ast} \\
z
\end{pmatrix}
\]

\[
\alpha^{\ast}\alpha - \beta^{\ast}\beta = 1
\]

(Bogoliubov SU(1,1) transformation)
- Coupling of FQHE states to static extrinsic geometry was studied by Wen and Zee (1992) in connection with the "shift" in FQHE states on the sphere (a popular finite-size geometry). There is a coupling to Gaussian curvature $K$ through a gauge-field they called (with a slight abuse of terminology) the "spin-connection".

$$K = \varepsilon^{ab}_\alpha \partial_a \Omega_b$$

$$S = \int d^2r dt A_\mu J^\mu + \Omega_\mu J^\mu_s - \frac{\hbar K_{ij}}{4\pi} \varepsilon^{\mu\nu\lambda} a^i_\mu \partial_\nu a^j_\lambda$$

2D orbital spin quantized to be integer or half-integer by Gauss-Bonnet

$$J^\mu = \frac{q_i e}{2\pi} \varepsilon^{\mu\nu\lambda} \partial_\nu a^i_\lambda$$

$$J^\mu_s = \frac{s_i}{2\pi} \varepsilon^{\mu\nu\lambda} \partial_\nu a^i_\lambda$$
• Wen and Zee, and later Read (2009), interpreted the 2D “spin” that couples to curvature as a property related to local isotropy under rotations about the normal the curved 2D surface embedded in 2D Euclidean space, and concluded it lost its meaning when such “rotational invariance” was broken by impurities.

• This turns out to be incorrect. 2D spin quantization is a topological property (Gauss-Bonnet) of an incompressible state because the number of independent states of the composite boson on a compact manifold is modified by the curvature, but must still be an integer. This does not require that a unitary symmetry such as rotation symmetry is maintained.
• The metric I have introduced is quite different from the induced metric due to extrinsic embedding considered by Wen and Zee. It is also dynamic.

• The conserved Gaussian curvature current is

\[ J^\mu_g = \varepsilon^{\mu\nu\lambda} \partial_\nu \Omega_g \Omega^\lambda \]

• The electronic charge density (for a single-component FQHE state) is now

\[ J^0 = \sigma_H B + \frac{e^*}{2\pi} \bar{s}K \]

\[ \sigma_H = \frac{pe^2}{2\pi q\hbar} \]

\[ e^* = \frac{e}{q} \]

Gaussian curvature

elementary fractional charge
• This is very similar to the quantum Hall ferromagnet, where the electrons charge density responds to both the magnetic flux density and the Berry curvature of the ferromagnetic order parameter.

• Here the excess charge relative to the Hall response is proportional to the Gaussian curvature of the metric.

\[ K = -\frac{1}{2} \partial_a \partial_b g^{ab} + \frac{1}{8} (\partial_a g^{ce})(\partial_b g^{df}) \epsilon^{ab} \epsilon_{cd} g_{ef} \]

Brioschi formula

\[ \delta \rho_q = \frac{e^* s}{4\pi} q_a q_b g_q^{ab} \]

This will reproduce the Girvin et al result that the guiding-center structure function is \( O(q^4) \)
Guiding-center structure function

\[ S(q) \]

\[ S_\infty \]

\[ \rho(q) = \sum_i e^{iq \cdot R_i} \]

\[ \propto q^4 \]

- crucial features at \( q = 0, \infty \)

\[ \langle \rho(q) \rangle = 2\pi \nu \delta^2(q \ell_B) \]

\[ \langle \rho(q) \rho(q') \rangle - \langle \rho(q) \rangle \langle \rho(q') \rangle = 2\pi S(q) \delta^2(q \ell_B + q' \ell_B) \]
- correlation energy per magnetic area

\[
\frac{E_0}{N_\Phi} = \int \frac{d^2 q}{2\pi} v(q) S(q) \frac{N}{N_\Phi} = \nu
\]

- \(S(q)\) is the ground-state structure factor, which describes the zero-point fluctuations of the \(R_i\)

- make an area-preserving shear deformation of the ground state, i.e., of \(S(q)\); the energy increase will be \textbf{quadratic} in the deformation

\[
q_a \rightarrow (\delta_a^b + \lambda_{ac} \epsilon^{cb}) q_b
\]

\[
\frac{\delta E}{N_\Phi} = \frac{1}{2} G^{abcd} \lambda_{ab} \lambda_{cd}
\]

\text{symmetric}

\text{guiding-center shear modulus}
Motivated by Feynman’s theory of the roton in $^4$He, GMP used the single-mode approximation as a variational ansatz for the collective excitation:

$$\Psi_{sma}(q) = \rho(q)\Psi_{0}^{exact}$$

$$\varepsilon(q) = E(q) - E_{0}^{exact} = \frac{A(q)}{S(q)}$$

$$A(q) = \frac{1}{2} \int \frac{d^2q'\ell_B^2}{2\pi} \nu(q') (S(q' + q) + S(q' - q) - 2S(q')) (2\sin \frac{1}{2} q \times q' \ell_B^2)^2$$

GMP did not interpret $A(q)$, but evaluated it numerically, using the Laughlin state $S(q)$ obtained by Monte Carlo methods.

$\lim_{\lambda \to 0} A(\lambda q) \to \lambda^4 G^{abcd} q_a q_b q_c q_d \ell_B^4$
• In the long wavelength limit, the GMP result can be written as
\[ \varepsilon_{\text{exact}}^{(q)} S(q) \leq G_{abcd}^{a'b'} q_{a'b'} q_{c'd'} \ell_B^4 \]

• This turns out to be an equality for systems with a single collective mode (single-component FQHE states, as well as Wigner-lattice states with one electron per unit cell)

• As GMP recognized, if the collective mode is gapped (i.e., the state is incompressible), \( S(q) \) must be quartic at long wavelengths. This was their fundamental insight into FQHE incompressibility.
• From this we learn that the fundamental stiffness of the incompressible FQHE states is their resistance to area-preserving distortions that change the shape of the correlation hole around a guiding center from the shape that minimizes the energy.

• The collective degrees of freedom can be described as one (or more) **UNIMODULAR** positive definite real-symmetric spatial/metric tensor fields

\[
g^{(\alpha)}_{ab}(r, t) = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \quad \text{det } g = 1
\]

As a quadratic form, this describes a local “shape of a circle”
Laughlin is like Mott-Hubbard + shape variable!

- Physically, it is the shape of the “attached flux” of the “composite bosons” that condense in the Chern-Simons “flux attachment picture:

  region with 3 flux quanta surrounding the electron. Other electrons are excluded from this region (analogy is a Hubbard model lattice site)

  at 1/3 filling, an electron with 3 “attached” flux quanta behaves like a neutral boson

area-preserving shape deformation of the exclusion region costs correlation energy
• Action: one component, filling p/q

\[ S_0 = \frac{\hbar}{2\pi} \int d^3x \epsilon^{\mu\nu\lambda} \left( \frac{q}{2} a_\mu \partial_\nu a_\lambda + \left( \frac{peA_\mu}{\hbar} + \bar{s}\Omega_\mu \right) \partial_\nu a_\lambda \right) \]

\[ S = S_0 - \int dt H \]

sum of geometry-dependent correlation energy and coulomb interaction between excess charge (Gaussian curvature)

\[ H = \int \frac{d^2r}{2\pi \ell_B^2} u(g) \]

\[ + \frac{1}{2} \int d^2r \int d^2r' V(r, r') K(r) K(r') \]

To quadratic order:

\[ e^* \bar{s}B[\omega_a(r), \omega_b^*(r')] = 2\pi i\hbar \delta^2(r - r') \]

\[ \epsilon^{ab} \omega_a^* \omega_b = i \]

\[ g_{ab} = \omega_a^* \omega_b + \omega_a \omega_b^* \]

\[ \Omega_\mu = \frac{1}{2} \delta_\mu^a \epsilon_{ab} \partial_c g^{bc} + A_\mu^g \]

\[ A_\mu^g = \epsilon^{ab} \omega_a^* \partial_\mu \omega_b \]

geometric gauge field
• Long wavelength behavior of the Girvin et al. single-mode approximation is exactly reproduced

• $S(q)$ is given by zero-point fluctuations the metric around its mean value; $q^4$ behavior given correctly.

• Gives “Hall viscosity” (Stress induced in fluid by electric field-gradients).

• Non-linear equations for structure of quasiparticle/quasiholes

• Generalization to multicomponent cases (cf “Jain/Hierarchy cases): one metric per condensate
“spin” = -1

Laughlin 1/3 state
p=1, q=3

- occupation pattern inside the “composite boson” or “elementary droplet” ("guiding center spin")
- geometry (shape) of the droplet
• example 1/3 Laughlin state:

\[
\begin{array}{cccccc}
1 & 3 & 5 & 7 \\
\frac{1}{2} & \frac{3}{2} & \frac{5}{2} & \frac{7}{2}
\end{array}
\]

Laughlin state

\[
\begin{array}{cccccccc}
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0
\end{array}
\]

uniform reference state

\[
\begin{array}{cccccccc}
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3}
\end{array}
\]

elementary droplet

\[s_0 = -1\]

\[L^z = \frac{1}{2} = \frac{3}{2} + s_0\]

\[L^z = \frac{3}{2}\]

• example: 2/3 "anti-Laughlin" state:

\[011011011011\]

\[s_0 = +1\]

Odd under particle-hole transformation!