When is a “wavefunction” not a wavefunction?
A quantum-geometric interpretation of the Laughlin state

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• The FQHE is a fundamentally problem of interacting “guiding centers” with a non commutative geometry: what is the true meaning of Laughlin’s “wavefunction”?

• An interpretation of “z” in the Laughlin “wavefunction” as the intrinsic geometry of flux attachment.

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In 1983 Bob Laughlin’s numerical diagonalization study of a system of 3 interacting electrons (!) in the lowest Landau level led him to his famous wavefunction.

\[ \Psi = \prod_{i<j} (z_i - z_j)^q \prod_i e^{-\frac{1}{2} z_i^* z_i} \]

It quickly became clear that this was the correct solution to the puzzle of the 1/3 FQHE and was the prototype model for the other FQHE states.

\[ z = \frac{x + iy}{\sqrt{2\ell_B}} \]
The Laughlin state was convincing because it explained why $\nu = 1/3$ was seen, but not $\nu = 1/2$.

Its validity was subsequently established by numerical exact diagonalization studies of the adiabatic evolution between a model system with a “short range pseudopotential” for which the Laughlin state was the **exact ground state**, and a realistic Coulomb interaction.

**FIG. 6.** Low-lying states at $N = 6, 2S = 15$ ($\nu = 1/3$) as the “hard-core” pseudopotential component $V_1$ is varied. The other $V_m$ take their Coulomb values. $V_3$ and the Coulomb value (C) of $V_1$ are marked. Angular momentum quantum numbers $L$ are indicated. Also shown is the projection of the LJ state on the ground state. In the gapless regime ($\lambda > 1.25$), the LJ state reappears as the **highest** $L = 0$ level.
• So it is known to work, but **why**? (In my opinion, this question was never satisfactorily answered)

a common rationalization:

“Laughlin’s wavefunction cleverly lowers the Coulomb correlation energy by placing its zeroes at the locations of the particles”

we will see that this is an empty statement
A key aspect of the Laughlin wavefunction is that it is a **holomorphic function** (times a non-holomorphic Gaussian).

This is the **lowest-Landau level** property:

$$a \psi(z, z^*) = 0$$

(annihilated by Landau-level lowering operator)

$$\psi(z, z^*) = f(z) e^{-\frac{1}{2} z^* z}$$

**holomorphic function**

**Gaussian**

$$a = \frac{1}{2} z + \frac{\partial}{\partial z^*}$$

$$a^\dagger = \frac{1}{2} z^* - \frac{\partial}{\partial z}$$

$$[a, a^\dagger] = 1$$

Schrödinger form of Landau level (harmonic oscillator) ladder operators.
Laughlin presented his state as a lowest-Landau-level wavefunction, and this interpretation seems to have been uncritically accepted. **But I will argue that its holomorphic character has NOTHING to do with the lowest Landau level (LLL)!**

Some evidence against the LLL interpretation:

- The Laughlin state also occurs in the second Landau level
- The Laughlin state occurs in models of Chern insulators with flat bands (no Landau levels)

(In fact, the Laughlin state should not even be interpreted as a Schrödinger wavefunction ....)
When there is Landau quantization, the classical (with commuting components) displacement of an electron splits up into a non-classical guiding center displacement plus the cyclotron-orbit radial vector:

\[ [\tau^a, \tau^b] = 0 \]

\[ \tau^a = \tilde{R}^a + \tilde{R}^a \]

guiding center of orbit

\[ [\tilde{R}^a, R^b] = -i\ell_B^2 \epsilon^{ab} \]

\[ [\tilde{R}^a, \tilde{R}^b] = 0 \] independent

area per London flux quantum = \(2\pi\ell_B^2\)
• The remaining degrees of freedom after Landau quantization are the non-classical guiding centers. They define a quantum geometry isomorphic to phase space, and obey an uncertainty principle.

\[ [R^a, R^b] = -i \ell_B^2 \epsilon^{ab} \]

• They cannot be described by a Schrödinger wavefunction! (only by a Heisenberg state)

The guiding centers are generic to any Landau level.

We must abandon the Schrödinger picture and use a Heisenberg description of the guiding center.
• In Laughlin’s “symmetric gauge” scheme, the guiding centers are described in a basis of circular states centered at an arbitrary origin.

• This circular shape is also the shape of the Landau orbits.

\[ \psi_{0m}(z, z^*) \propto z^m e^{-\frac{1}{2} z^* z} \]

\[ \bar{a}^\dagger = \frac{1}{2} z - \frac{\partial}{\partial z^*} \]
\[ a^\dagger = \frac{1}{2} z^* - \frac{\partial}{\partial z} \]
\[ \bar{a} = \frac{1}{2} z^* + \frac{\partial}{\partial z} \]
\[ a = \frac{1}{2} z + \frac{\partial}{\partial z^*} \]

Circular basis guiding center
Ladder operators
Landau orbit
Ladder operators

\[ a\psi_{00} = \bar{a}\psi_{00} = 0 \]
\[ |\psi_{m0}\rangle = \frac{(\bar{a}^\dagger)^m}{\sqrt{m!}} |\psi_{00}\rangle \]

Action on LLL states:
\[ \bar{a}^\dagger f(z)e^{-\frac{1}{2} z^* z} = z f(z)e^{-\frac{1}{2} z^* z} \]
The Heisenberg form of Laughlin is therefore an unentangled product of a correlated guiding-center state with a (trivial) harmonic oscillator state of the Landau orbits:

\[
|\Psi^q_L\rangle = \left( \prod_{i<j} (\bar{a}^\dagger_i - \bar{a}^\dagger_j)^q |\bar{\Psi}_0\rangle \right) \otimes |\Psi_0\rangle
\]

\[
\bar{a}_i |\bar{\Psi}_0\rangle = 0
\]

\[
\bar{a}_i |\Psi_0\rangle = 0
\]

at this point, we "purify" the Laughlin state by removing its LLL "baggage"
The Landau orbits are now gone: what defines $\bar{a}$ and $\bar{a}^\dagger$?

$$[L, \bar{a}^\dagger_i] = \bar{a}^\dagger_i$$

$$L = \frac{1}{2} \sum_i \left( \bar{a}_i^\dagger \bar{a}_i + \bar{a}_i \bar{a}_i^\dagger \right)$$

$$= \frac{1}{2\ell^2_B} \sum_i \bar{g}_{ab} R_i^a R_i^b$$

The guiding center metric $g_{ab}$ that defines the basis of guiding-center ladder operators is an **INDEPENDENT** variational geometric parameter of the Laughlin state, that must be chosen to minimize the correlation energy.

- **not necessarily congruent!**
• the guiding-center metric is a parameter of the pseudopotential Hamiltonian that can be used to define the Laughlin state:

\[
H(g) = \int \frac{d^2 q \ell_B^2}{2\pi} \sum_m V_m L_m (q_g^2 \ell_B^2) e^{-\frac{1}{2} q_g^2 \ell_B^2} \sum_{i<j} e^{i \vec{q} \cdot (\vec{R}_i - \vec{R}_j)}
\]

• It is the **shape** of the composite boson formed by “flux attachment”

“Elementary droplet” or “composite boson” of 1/3 Laughlin state:
central orbital filled, next two empty

area-preserving zero-point fluctuations of shape give \( q^A \) structure factor
• In general

\[ a = \frac{1}{2} z + \frac{\partial}{\partial z^*} \]
\[ a^\dagger = \frac{1}{2} z^* - \frac{\partial}{\partial z} \]

\[ \bar{a} = \frac{1}{2} \bar{z} + \frac{\partial}{\partial \bar{z}^*} \]
\[ \bar{a}^\dagger = \frac{1}{2} \bar{z}^* - \frac{\partial}{\partial \bar{z}} \]

Landau orbits congruent to \( |z| = \text{constant} \)
guiding-center flux attachment shape congruent to \( |\bar{z}| = \text{constant} \)

\[ \bar{z} = \alpha z^* + \beta z \]
\[ |\alpha|^2 - |\beta|^2 = 1 \]

• “zeroes of wavefunction” only coincide with locations of particles if \( \beta = 0 \).
• So where do the holomorphic functions come from?

coherent states of the guiding centers are non-orthogonal and overcomplete:

$$\bar{a} | \bar{z} \rangle = \bar{z} | \bar{z} \rangle$$

$$\langle \bar{z} | \bar{z}' \rangle \equiv S_g(\vec{r}, \vec{r}')$$

complex positive-indefinite Hermitian overlap matrix that depends implicitly on the metric
• The overlap defines a “quantum geometry”
• A “quantum distance” measure is defined by
  \[ d(\vec{r}, \vec{r}')^2 = 1 - |S(\vec{r}, \vec{r}')| \]
• The non-null eigenfunctions of the overlap define an orthonormal basis:

\[
\int \frac{d^2 r'}{2\pi \ell_B^2} S_{g}(\vec{r}, \vec{r}') \psi_\lambda(\vec{r}') = s_\lambda \psi_\lambda(\vec{r})
\]

| \Psi_\lambda \rangle = \frac{1}{\sqrt{s_\lambda}} \int \frac{d^2 r}{2\pi \ell_B^2} \psi_\lambda(\vec{r}) | \bar{z}(\vec{r}) \rangle
• For the guiding-center coherent states, the non-null eigenstates of $S$ are degenerate, with the form $f(\bar{z}^*) e^{-\frac{1}{2} \bar{z}\bar{z}^*}$ holomorphic!

• Coherent-state representation of Laughlin state:

$$|\Psi\rangle \propto \prod_i \int \frac{d^2 r_i}{2\pi \ell_B^2} \left( \prod_{i<j} (\bar{z}_i^* - \bar{z}_j^*)^q \prod_i e^{-\frac{1}{2} \bar{z}_i \bar{z}_i^*} \right) \prod_{i} \bar{z}_i^* \prod_{i<j} (\bar{z}_i^* - \bar{z}_j^*)^q \prod_i e^{-\frac{1}{2} \bar{z}_i \bar{z}_i^*} |\{\bar{z}_i\}\rangle$$

Laughlin “wavefunction”, with replacement $z \rightarrow \bar{z}^*$ unchanged when composite boson has same shape as Landau orbit! $($$\bar{z} = z^*$$)$
The metric is the true collective degree of freedom of the FQHE state: it adjusts to non-uniform electric fields. Its curvature from spatial nonuniformity provides an additional gauge field similar to Berry curvature in quantum Hall ferromagnets.

gives
“Area” (perimeter) term in “momentum polarization”

gives
cut

relation to Hall viscosity through edge currents produced by entanglement cuts
\[ V(x) \]

- near edges:

fluid is compressed at edges by creating Gaussian curvature

\[ \delta J^0_e = \frac{e^* s}{2\pi} J^0_g \]

fluid density fixed by flux density

For larger \( s \), fluid becomes more compressible (less distortion needed for a given density change)

\[ g = \begin{pmatrix} \alpha(x) & 0 \\ 0 & \frac{1}{\alpha(x)} \end{pmatrix} \]

\[ J^0_g = -\frac{1}{2} \frac{d^2}{dx^2} \frac{1}{\alpha(x)} \]

Wednesday, March 20, 13
• On Chern Insulator “flat bands”, get a similar structure by projecting lattice-site orbitals into a single band, and renormalizing back to unit weight:

![Diagram](image)

• The overlap matrix encodes the quantum geometry in this case too, shows difference between topological and non-topological bands
summary

• The variable “z” in the Laughlin “wavefunction” depends on guiding center geometry, not Landau orbit shape, which are chosen to be congruent in Laughlin’s treatment

• The Laughlin “wavefunction” is really a guiding-center coherent-state amplitude