“Hall viscosity”, edge state dipole moment, and incompressibility of FQHE fluids.

F. D. M. Haldane, Princeton University

- Dissipationless “Hall viscosity” and the stress in incompressible Hall fluids with a non-uniform drift velocity field

- Stress anomaly and electric dipole moment at fluid edges.

- An unexpected relation to the “guiding center structure factor” and incompressibility.

arXiv:0906.1854v1

supported in part by DOE DE-SC0002140
quantum geometry (high field limit)

some algebras:

- guiding center algebra
  \[ [R_i^a, R_i^b] = -i\ell^2 \epsilon^{ab} \delta_{ij} \]

- guiding center density
  \[ \rho_q = \sum_i e^{iq_a} R_i^a \]
  \[ [\rho_q, \rho_{q'}] = 2i \sin \left( \frac{1}{2} \epsilon^{ab} q_a q_b \ell^2 \right) \rho_{q+q'} \]

- guiding center momentum
  \[ P_a = \frac{\hbar}{\ell^2} \epsilon_{ab} \sum_i R_i^b \]
  \[ [P_a, P_b] = -i\hbar^2 \epsilon_{ab} \rho_0 \]

- guiding center deformation
  \[ \Lambda^{ab} = \frac{1}{4\ell^2} \sum_i \{R_i^a, R_i^b\} \] ("squeezing")
  \[ [\Lambda^{ab}, \Lambda^{cd}] = \frac{1}{2} i (\epsilon^{ac} \Lambda^{bd} + \epsilon^{ad} \Lambda^{bc} + \epsilon^{bc} \Lambda^{ad} + \epsilon^{bd} \Lambda^{cd}) \] SO(2,1) Lie Algebra

linear transformations (rotations, stretch, shear,...)
Collective mode in the FQHE model with short-range interactions (\(V_1\) pseudopotential only) at 1/3 filling (Laughlin state is exact ground state)

\[ \Delta E(q) \]

"roton" (2 quasiparticle + 2 quasiholes)

\[ f(q) \propto q^4 \]

Variational bound (Girvin et al. 1985)

\[ \Delta E(q) \leq \frac{f(q)}{S_0(q)} \]

\[ S_0(q) = \frac{1}{A} \left( \langle \rho_q \rho_{-q} \rangle - \langle \rho_q \rangle \langle \rho_{-q} \rangle \right) \]

"guiding-center structure factor" \(S_0(q)\) must also vanish as \(q^4\) for gap to be finite.
only case where limiting behavior was previously explicitly known is $1/m

Laughlin state:

$$S_0(q) \rightarrow \frac{1}{2\pi \ell^2} \left( \frac{m - 1}{2m} \right) \left( \frac{q^2 \ell^2}{2} \right)^2$$

(Girvin et al, 1985 from mapping to scale-invariant one-component plasma, and using compressibility sum rule)

$q^2 \equiv g^{ab} q_a q_b$

Note: Laughlin state is rotationally invariant with

$$L^z = g_{ab} \Lambda^{ab}$$

Euclidean metric, defines rotational invariance

$\Psi_m(g^{ab})$ (Laughlin state is a continuous function of the metric, which is a true variational parameter)
• Although most studies assume rotational invariance, it is not a required or generic property of FQHE states.

• e.g., “tilting” the B-field removes rotational invariance
I will describe a remarkable connection of the coefficient of the $q^4$ small-$q$ behavior of the guiding center structure factor to the “Hall viscosity” that characterizes stress induced in a Hall fluid by a non-uniform electric field.

To bring out the structure, I will NOT assume rotational invariance at first.

This needs a “covariant” notation with upper indices for real space vectors, lower indices for reciprocal space vectors (there is no notion of orthogonality without a metric).
This derivation involved the edge. Can we get a “bulk” derivation of $\sigma_{ab}$ even though it is no longer defined by the rotationally-invariant-system formula

$$\sigma_{ab} = \sigma_{ac} g_{cb} = \frac{1}{A} \Psi \delta H \delta g_{ab} \Psi \Psi.$$ 

Yes, it can be derived from the continuity relation $V(r)$ violates momentum conservation

$$\pi_a = \frac{\hbar}{\ell^2_B} \epsilon_{ab} \sum_{i=1}^{N} (e^{i \frac{1}{2} q \cdot R_i}) R^b_i (e^{i \frac{1}{2} q \cdot R_i}).$$

Fluid dynamics of the incompressible state

$$H = \int d^2 r \ (h_0(r) + V(r) \bar{\rho}(r))$$

(qquasi)-local translationally-invariant Hamiltonian that gives rise to incompressibility

Slowly-varying “external” potential (including Hartree potential from long-range part of Coulomb force).

continuity equation for particle density

$$\partial_t \rho + \nabla_a J^a = 0$$

definition of flow velocity field

$$J^a = \rho v^a$$

Drift velocity field

$$v^a(r) = \frac{\ell^2_B}{\hbar} \epsilon^{ab} \nabla_b V(r)$$

V(r) violates momentum conservation

stress tensor

$$\pi_a = \int \frac{d^2 q}{(2\pi)^2} e^{-i q \cdot r} \langle \tilde{\pi}_a(q) \rangle$$

momentum density
Viscosity (linear response)

\[ \sigma_{ab} = p \delta_{ab} + \eta_{bd} \nabla_c v^d + O(v^2) \]

- **Stress tensor**
- **force across boundary:**
  \[ dF_a = \sigma_a^b dA_b \]
- **area element**

**Stress tensor**

**There is no “symmetry” of the Stress tensor because it has one (position-type) upper index and one (momentum-type) lower index**

**IF rotational invariance with metric \( g_{ab} \) is present**

\[ \sigma^{ab} \equiv g^{ac} \sigma_c^b = \sigma^{ba} \text{ (symmetric)} \]

- **special feature of an incompressible Hall fluid:**
  \[ p = 0 \]
  - **vanishing hydrostatic pressure!**
- apply pressure to outer edge

- the edge current increases, generating a balancing force that compensates the applied “pressure”

- no transmission of force to interior (unlike a classical incompressible fluid which is gapless and transmits pressure)
• rewrite viscosity tensor as a dimensionless rank-4 tensor with 4 upper indices:

\[ \eta^{ac}_{bd} = \frac{\hbar}{\ell_B^2} \epsilon_{be} \epsilon_{df} \Gamma^{acef}_{bd} \]

\[ \nu^d_{\text{drift}} = \frac{\ell_B^2}{\hbar} \epsilon^{de} \nabla_e V \]

Dimensions of viscosity

\( \Gamma \) is symmetric in the first pair and in the second pair of indices:

\[ \eta^{ac}_{ad} = 0 (p = 0) \]

No dissipation:

\[ \sigma^a_b \nabla_a J^b = 0 \]

\[ \eta_{bd} = -\eta_{ca} \]

“Hall viscosity”

\[ \Gamma_{abcd} = -\Gamma_{cdab} \]

“Dimensionless Hall viscosity”
• after a little work, I obtain

$\Gamma^a_{bcd} = \frac{1}{N_{\text{orb}}} \langle \Psi_0 | \frac{1}{2i} [\Lambda^{ab}, \Lambda^{cd}] | \Psi_0 \rangle$

$\Lambda^{ab} = \frac{1}{4\ell^2_B} \sum_{i=1}^{N} \{ R^a_i, R^b_i \} = \Lambda^{ba}$

$[\Lambda^{ab}, \Lambda^{cd}] = \frac{i}{2} (\epsilon^{ac} \Lambda^{bd} + \epsilon^{ad} \Lambda^{bc} + \epsilon^{bc} \Lambda^{ad} + \epsilon^{bd} \Lambda^{ac})$

• This is the SO(2,1) Lie algebra (like the Lorentz group in 2+1 dimensions)

• It has three generators $\Lambda^{11}, \Lambda^{21}$ and $\Lambda^{22}$. Casimir is

$C_2 = - \det |\Lambda| = (\Lambda^{12})^2 + \left( \frac{\Lambda^{11} - \Lambda^{22}}{2} \right)^2 - \left( \frac{\Lambda^{11} + \Lambda^{22}}{2} \right)^2$

number of electron orbitals $N_{\text{orb}} = \frac{A}{2\pi \ell^2}$
Structure factor:
new result (without invoking rotational invariance):

$$\lim_{\lambda \to 0} S_0(\lambda q) = \frac{\lambda^4}{4} \Gamma_{S}^{abcd} q_a q_b q_c q_d + O(\lambda^6)$$

$$\Gamma_{S}^{abcd} = \frac{1}{N_{\text{orb}}} \left( \langle \Psi_0 | \frac{1}{2} \{ \Lambda^{ab}, \Lambda^{cd} \} | \Psi_0 \rangle - \langle \Psi_0 | \Lambda^{ab} | \Psi_0 \rangle \langle \Psi_0 | \Lambda^{cd} | \Psi_0 \rangle \right)$$

Combines naturally with Hall viscosity

$$i \Gamma_A^{abcd}$$

$$\Gamma_{S}^{abcd} + i \Gamma_A^{abcd} = \frac{1}{N_{\text{orb}}} \left( \langle \Lambda^{ab} \Lambda^{cd} \rangle - \langle \Lambda^{ab} \rangle \langle \Lambda^{cd} \rangle \right)$$

can write as a positive
3x3 Hermitian matrix

$$M_{(ab),(cd)} = (M_{(cd),(ab)})^*$$

(ab) = (11), (12), (22)
Hall viscosity can also be represented as a symmetric (definite) rank-2 tensor:

\[ \Gamma^{abcd} = \Gamma^{A}_{abcd} = 2\pi \frac{1}{2} \left( \epsilon^{ac} Q^{bd} + \epsilon^{ad} Q^{bc} + \epsilon^{bc} Q^{ad} + \epsilon^{bd} Q^{ac} \right) \]

A symmetric rank-2 tensor

- Read (2009) defines a scalar Hall viscosity of rotationally-invariant Hall fluids as

\[ \eta^{(A)} = \frac{1}{2} \rho \bar{\ell} z \]

fluid density

- then

\[ Q^{ab} = \eta^{(A)} \frac{\ell^2}{\hbar} g^{ab} \]

constant in an incompressible region with translational invariance

metric defined by rotational invariance

Avron et al. result:

\[ Q^{ab} = \left( \frac{1}{4\pi} \sum_n \nu_n s_n \right) g^{ab} \]

filling of \( n^{th} \) Landau level

Galilean metric (from mass tensor)

intrinsic \( L^z \) angular momentum per particle
significance of $Q^{ab}$:

- Discontinuity of $Q^{ab}$ across boundary means stress force from I to II does not balance that from II to I! ($\nabla_a \nabla_b V$ is continuous)

- Get an intrinsic dipole moment at the boundary so the stress anomaly is balanced by the force

\[
dF_a = dP_b \nabla_b E_a \\
(eE_a = -\nabla_a V)
\]

\[
dp^a = e\Delta Q^{ab} \epsilon_{bcd} dL^c
\]

- $\Delta Q^{ab} = Q^{ab}_I - Q^{ab}_{II}$

- Static boundary (must be an equipotential = a flow line)

- $dL \cdot \nabla V = 0$
specialize to rotationally invariant case

- elementary droplet of Hall fluid has $p$ particles in $q$ orbitals and angular momentum $L^z = \frac{1}{2}pq - (s + s')$

- In the high-field limit, $-s$ is the total Landau-orbit angular momentum of the droplet, and $-s'$ is the intrinsic guiding center angular momentum (a modified “shift”), (both are quantized)

- $s'$ is odd under particle-hole conjugation of a Landau level, and vanishes when all Landau levels are filled or empty (Integer QHE case).
For the guiding-center structure factor

\[ S_0(q) \rightarrow \frac{1}{2\pi \ell^2} \gamma \left( \frac{q^2 \ell^2}{2} \right)^2 \gamma \geq \frac{|s'|}{q} \]

- for the known results for the Laughlin states, the bound is an equality.

- Numerical results for the Moore-Read state (adiabatic variation of periodic bc) shows that the Hermitian matrix \( M_{\{ab,cd\}} \) has a single eigenvalue, even for sizes too small for convergence to the quantized value given by the (modified) shift in rotationally-invariant geometries, showing that the bound is satisfied as an equality for model wavefunctions derived from conformal field theory.

- This is not true when corrections due to e.g. Coulomb interactions are included.
Two possibilities for the generic rotationally-invariant models

• the bound is an inequality, because the RPA-like dressing of the ground state by zero-point fluctuations of the collective mode make the system more compressible than the ideal model cft reference wavefunction.

• or, perhaps, the bound becomes an equality again in the thermodynamic limit (not ruled out yet).
relation of edge dipole to “shift”

The dipole at a segment of the edge has a momentum

\[ dP_a = \frac{\hbar}{e \ell_B^2} \epsilon_{ab} dp^b \]

momentum dipole

doesn’t contribute to total momentum:

\[ \int dP_a = 0 \]

it does contribute an extra term to total angular momentum:

\[ \Delta L^z (g) = \hbar \oint \epsilon^{ab} g_{bc} r^c dP_a \neq 0 \]
summary

- add momentum continuity equation to supplement charge continuity of incompressible Hall fluids. Rotational invariance not assumed.
- linear response of stress tensor to non-uniform drift velocity
- unexpected relation to FQHE structure factor, gives bound that is equality for cft model wavefunctions.
- significant of SO(2,1) deformation algebra.
- more in arXiv: 0906:1854