Generalized Pauli principle for Read-Rezayi non-Abelian Quantum Hall States

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Selection rule on “admissible configurations” in Laughlin, Moore-Read and Read-Rezayi states

Numerical studies

Effects on entanglement spectrum

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Collaborations with Emil Prodan, Andrei Bernevig, and Hui Li
Laughlin FQHE state

\[ \Psi = \Phi(z_1, z_2, \ldots, z_N) \prod_{i=1}^{N} e^{-\varphi(r_i)} \]

- \( \nu = 1/m \) Laughlin state
- "occupation number"-like representation in orbitals \( z^m \), \( m = 0,1,\ldots, N-1 \)

\[ \Phi(z_1, z_2, \ldots, z_N) = \prod_{i<j} (z_i - z_j)^m \]

\[ \nabla^2 \varphi(r) = \frac{2\pi B(r)}{\Phi_0} \]

- lowest Landau level
- \( N \)-variable (anti)symmetric polynomial

This is the "dominant" configuration of the Laughlin state
“Dominance”

- convert occupation pattern to a **partition** \( \lambda \), “padded” with zeroes to length \( N \):
  - \( 1001001 \rightarrow \lambda = \{ \lambda_1, \lambda_2, \lambda_3 \} = \{6, 3, 0\} \)
- \( \lambda \) dominates \( \lambda' \) if
  - \( |\lambda| \equiv (\sum_i \lambda_i) = |\lambda'| = M \)
  - \( (\sum_{j \leq i} \lambda'_{j}) \leq (\sum_{j \leq i} \lambda_{j}) \) for all \( i = 1, 2, ..N-1 \)
“dominance” and “squeezing”

* (pairwise) squeezing: move a particle from orbital $m_1-1$ to $m_1$ and another from $m_2+1$ to $m_2$ where $m_1 \leq m_2$.

1001001001001001001...1001 \quad A
1000101001001010001...1001 \quad B

A dominates B \hspace{1cm} (A > B)

* dominance is a *partial ordering*: if $A > B$ and $B > C$, then $A > C$. 
• When expanded in occupation number states, the (polynomial) \( 1/m \) Laughlin state only contains configurations dominated by the most compressed (minimum \( M \)) \( (1,m) \)-admissible configuration” where no group of \( m \) consecutive orbitals contains more than 1 particle.

• “Admissibility” can be thought of as a generalized Pauli principle.
Compactification of the Lowest Landau level on the Riemann sphere.

Identify orbitals $m = 0, 1, ..., N\phi$ with orbitals $L_z = S, S-1, ..., -S$ on a sphere enclosing magnetic monopole charge $N\phi = 2S$.

Uniform QHE states are rotationally-invariant, $L_{\text{tot}} = 0$. 
Beyond “standard” occupation number formalism

• \( k \)-particle \( \frac{1}{m} \) Laughlin droplet creation operator (circular droplet centered at \( R \)):

\[
\eta_{km}(R)\dagger |vac\rangle \propto \prod_{i>j} (z_i - z_j)^m \prod_{i=1}^{k} \psi_R(r_i)
\]

• For \( k = 1 \), (\( m \) has no meaning in this case), this is just the standard lowest Landau-level single-particle creation operator

\[
c(R)\dagger |vac\rangle \propto \psi_R(r_1)
\]
Read-Rezayi (includes Laughlin, Moore-Read)

FQHE states are defined by

\[ \nu = \frac{k}{km + 2} \]

\[ \eta_{k+1,m}(R) |\Psi\rangle = 0 \]

\[ \eta_{2,m'}(R) |\Psi\rangle = 0, \quad m' \leq m \]

\[ \eta_{k,m}(R_i) |\Psi\rangle = 0 \]

“Admissible” configurations:

Not more than k particles in km+2 consecutive orbitals
For m > 0, not more than one particle in m consecutive orbitals

for all \( R \)

at locations \( R_i \)

of pinned elementary quasiholes
• On the sphere, the number of charge \(-e/(km+2)\) elementary quasiholes for a given \(N\), \(N_\Phi\) is

\[
N_{qh} = k(N_\Phi - \frac{1}{2}mk(k - 1)) - (km + 2)(N - k)
\]

• The size of the basis set of quantum states (with unpinned quasi holes) is equal to the number of admissible configurations.

• The states can be completely constructed out of configurations dominated by the dominant admissible configuration ("top" configuration).

• These are a very small subset of lowest Landau level states!
Jack Polynomials.

• For \( m = 0 \) (bosonic case) the \( \nu = k/2 \) Read–Rezayi multi–quasihole states are spanned by the set of Jack symmetric polynomials \( J_{\lambda}^{-k+1}(z_1,...z_N) \) with an admissible partition \( \lambda \), which form a complete but non–orthogonal basis. (See Feigin, Miwa, Jimbo and Mukhin 2002, and Bernevig and Haldane, cond–mat/0707.3637)

• \( J^{\alpha}_{\lambda}(z) \) with parameter \( \alpha \) real positive and unrestricted are orthogonal polynomials; here \( \alpha \) is in general negative rational, and \( \lambda \) are restricted to “admissible” partitions.
Fermionic $2/4 = 1/2$ Moore-Read state

uniform vacuum state on sphere:

110011001100110011001100110011001

even fermion number $-e/2$ double quasihole ($h/e$ vortex) at North Pole:

011001100110011001100110011001100

odd fermion number $-e/2$ double quasihole ($h/e$ vortex) at North Pole:

10011001100110011001100110011001

fractionalization: one $-e/4$ quasihole ($h/2e$ vortex) at North Pole, one near equator.

101010101010101001100110011001100

These translate into explicit wavefunctions that can be calculated in finite-size systems
3/5 (fibonacci) Read-Rezayi state primary configurations

<table>
<thead>
<tr>
<th>Configuration</th>
<th>Elementary -e/5 Vortex at North Pole</th>
<th>Vortex Moves by Hopping 5 Orbitals at a Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>11100111001110011100111001110011100...</td>
<td></td>
<td></td>
</tr>
<tr>
<td>110101101011010110101101101101...</td>
<td></td>
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</tr>
<tr>
<td>110011110011100111001110011100111001...</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>0111001110011100111001110011100111...</td>
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</tr>
<tr>
<td>010110101101011010110101101011010...</td>
<td></td>
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<tr>
<td>100111001110011100111001110011100111...</td>
<td></td>
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</tr>
<tr>
<td>0110101101011010110101101011011010...</td>
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</tr>
</tbody>
</table>

For charge -ne/5, n > 1 there are always 2 orthogonal primary states.
explicit numerical calculations

- **Strategy:** obtain full set of highest-weight states by solving

\[ L^+_{tot} |\Psi\rangle = 0 \]

- The number of admissible configs at each \( L_z \) tells us how many we need. We exclude from the basis set configs not dominated by the dominant admissible config. This gives a highly overdetermined system of equations!

- Within the full basis set thus obtained, impose the condition that pins the quasiholes at the desired locations.
Partial ordering of occupation number configurations with fixed $L_z$

- squeezing decreases the variance

\[
\sum_{m=0}^{m_{\text{max}}} m^2 n_m - \left( \sum_{m=0}^{m_{\text{max}}} m n_m \right)^2
\]

- “Top”
- “admissible”
- “squeezed from admissible”
- “Bottom”
- “excluded”

\[ P_0 | \text{“excluded config.”} \rangle = 0 \]
key point:
Null space is invariant under the Euclidean group

- Disk: \([P_0, a] = 0\)
- Sphere: \([P_0, L^+] = 0\)

- Use Wigner-Eckert: need to (simultaneously) solve
  \[L^+|\Psi\rangle = 0 \text{ and } P_0|\Psi\rangle = 0\]

- In the full basis this is an undetermined problem
  (more columns than rows)

- After “excluded” states are removed, it is
  overdetermined (more rows than columns)!

- (can efficiently solve with a variant Lanczos-type technique to
  full floating-point accuracy.)
example:
16 electrons on sphere, maximum ν=1/2 Moore-Read density, plus 2h/e extra flux (single qubit when vortices are fixed)

16 spinless fermions on the sphere with 32 orbitals:
full basis: 601080390 projected basis: 825 (summed over Lz)

Lz = 2: find 6 zero modes of a sparse 5,800,384 x 6,170,810 overdetermined matrix (52x10⁶ non-zero matrix elements)

601,080,390 lowest LL states
825 MR null-mode states, of which 57 are highest weight
two Moore-Read vortices (fused)

FIG. 4: The particle density for fused probes as function of distance from the fusing point. Left/right panel refers to odd/even number of electrons. On the left, the different curves correspond to $N/N_\phi=9/16$, 11/20, 13/24, 15/28 and, on the right, the different curves correspond to $N/N_\phi=10/18$, 12/22, 14/26, 16/30.

\[
\begin{align*}
100110011001100110011 & \ldots \\
0110011001100110011001100 & \ldots
\end{align*}
\]

unpaired electron at North pole
Monodromy

- Hold one vortex at the north pole, and move the other in infinitesimal loops to map out the Berry curvature, in the two cases of even and odd fermion number.

- Integrate the berry curvature inside a closed path to get the monodromy.

![Graph showing Berry curvature for different cases](image)

FIG. 10: (Color online.) The Berry curvature obtained by moving one anyon while keeping the other fixed. Left-upper panel shows the results for odd number of electrons: $N/N_\phi=9/16$, $11/20$, $13/24$, $15/28$ and the right-upper panel shows the results for even number of electrons: $N/N_\phi=10/18$, $12/22$, $14/26$, $16/30$. The lower-left and lower-right panels show the sum and the difference between the odd and even results, respectively. For example, we added and subtracted the result for $N/N_\phi=10/18$ and $N/N_\phi=9/16$, and then the results for $N/N_\phi=12/22$ and $N/N_\phi=11/20$, etc.
for a path with a large radius, the relative Berry phase factor between the even and odd fermion number cases approaches -1 (as predicted!)

FIG. 11: (Color online.) The Berry phase accumulated by an anyon when moved along a path $\theta=\text{const}$, with the other anyon fixed at the North pole. The Berry phase is plotted against the area enclosed by the paths. Each curve is marked with the corresponding $N/N_\phi$ numbers. The insets show the electron density for the fused anyons, computed in Fig. 4, which one can use, experimentally, to distinguish between even/odd cases.
4 well-separated vortices (a qubit)

Note that the two state have slightly different “interference ripple” patterns in the electron density that will be exponentially small as the distance between the vortices increases, but which is a residual local physical difference between the states.
single-particle density

\[ m = 1 \] two-particle density

Tetrahedral arrangement of 4 MR $\hbar/2e$ vortices, (14 electrons, 28 orbitals)

One qubit is left after positions of vortices are fixed.

Sphere is mapped to unit disk.

the qubit doublet is split by the Coulomb interaction, both states are shown. THE SPLITTING AND LOCAL DIFFERENCE BETWEEN THE TWO STATES IS EXPECTED TO DISAPPEAR AS THE SYSTEM SIZE INCREASES.
four probes, tetrahedral pattern: candidate qubit pair, 14/28

zero-point motion of vortex positions

These are made with “STM + coulomb repulsion”: very close to the “exact” states!
non-Abelian Berry curvature, for increasing size (10-15 electrons)

as size increases, the (magnitude) of the non-abelian curvature field is seen to be concentrated near the quasiparticle cores, consistent with braiding. (For widely separated vortices, there should be vanishing non-abelian curvature in the regions in between the vortices, so the monodromy becomes purely topological)
Entanglement spectra and “dominance”

- Schmidt decomposition of Fock space into N and S hemispheres.
- Classify states by Lz and N in northern hemisphere, relative to dominant configuration. Lz always decreases relative to this (squeezing)
Represent bipartite Schmidt decomposition like an excitation spectrum (with Hui Li)

\[ |\Psi\rangle = \sum_{\alpha} e^{-\beta_\alpha/2} |\Psi_{N\alpha}\rangle \otimes |\Psi_{S\alpha}\rangle \]

- like CFT of edge states.
- A lot more information than single number (entropy)
  - many zero eigenvalues
    \[ e^{-\beta_\alpha} = 0 \]

FIG. 1: Entanglement spectrum for the 1/3-filling Laughlin states, for \( N = 10, m = 3, N_\phi = 27 \) and \( N = 12, m = 3, N_\phi = 33 \). Only sectors of \( N_A = N_B = N/2 \) are shown.
Look at difference between Laughlin state, entanglement spectrum and state that interpolates to Coulomb ground state.

FIG. 2: Entanglement spectrum for the ground state, for a system of $N = 10$ electrons in the lowest Landau level on a sphere enclosing $N_\phi = 27$ flux quanta, of the Hamiltonian in Eq. (12) for various values of $x$.

$$H = xH_c + (1 - x)V_1$$

$x=0$ is pure Laughlin.

Can we identify topological order in “physical as opposed to model wavefunctions from low-energy entanglement spectra?
Summary

- Generalized Pauli-like “admissibility” criterion gives counting of Laughlin/Moore-Read/Read-Rezayi states with quasi-holes, and specifies “dominant” configurations.

- Gives a simplified basis for practical calculations (analog of projection into the lowest Landau level, now into Read-Rezayi zero mode space)

- (Generalizes to describe quasiPARTICLES too, - with A. Bernevig)

- Entanglement spectra.